SOME CONSIDERATIONS ABOUT BORWEIN CONJECTURE

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Abstract. We try to make some contributions in proving the conjecture which P. Borwein established. In that order, we consider it in the matrix form and notice some wonderful relations. Also, we concentrate our attention to self-inversive polynomials and conclude that the whole conjecture can be written by three sequences of self-inversive polynomials. At last, our numerical evaluating persuade us that a few auxiliary conjectures are true We think that they can be useful in final proof.

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1. Introduction

Let us denote by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$$

and

$$J(n;q) = (q;q^3)_n \ (q^2;q^3)_n = \prod_{i=0}^{n-1}(1-q^{3i+1})(1-q^{3i+2}).$$

For fixed n, $J(n;q)$ is a monotonously decreasing function in the interval $[0,1]$ with a unique point of inflection $\xi \approx 0.448527$.

Peter Borwein in 1990. made the next conjecture.

Conjecture 1.1. *(***P. Borwein***) The polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ defined by*

$$J(n;q) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3), \tag{1.1}$$

have nonnegative coefficients.

Somewhere this conjecture denotes like the property $+--$.

Some similar conjectures are discussed by D.M. Bressoud [3].

Figure 1.a
The function $J(n;q)$ for $n=7$.

Figure 1.b
The coefficients of $A_n(q)$ $(n=10,11,12,13)$.

In the paper [2], G.E. Andrews has derived the next recurrence relations

$$\begin{aligned} A_n(q) &= (1+q^{2n-1})A_{n-1}(q) + q^n B_{n-1}(q) + q^n C_{n-1}(q), & A_0(q) &= 1,\\ B_n(q) &= q^{n-1}A_{n-1}(q) + (1+q^{2n-1})B_{n-1}(q) - q^n C_{n-1}(q), & B_0(q) &= 0,\\ C_n(q) &= q^{n-1}A_{n-1}(q) - q^{n-1}B_{n-1}(q) + (1+q^{2n-1})C_{n-1}(q), & C_0(q) &= 0. \end{aligned}$$

Notice that

$$\deg A_n(q) = n^2, \quad \deg B_n(q) = \deg C_n(q) = n^2-1 \quad (n>0).$$

If we write the Andrews' recurrence relations in the form

$$\begin{aligned} A_n(q) - A_{n-1}(q) &= q^n \left\{ q^{n-1}A_{n-1}(q) + B_{n-1}(q) + C_{n-1}(q) \right\},\\ B_n(q) - B_{n-1}(q) &= q^{n-1} \left\{ A_{n-1}(q) + q^n B_{n-1}(q) - qC_{n-1}(q) \right\},\\ C_n(q) - C_{n-1}(q) &= q^{n-1} \left\{ A_{n-1}(q) - B_{n-1}(q) + q^n C_{n-1}(q) \right\}, \end{aligned}$$

we see that the polynomials $A_n(q) = \sum_{j=0}^{n^2} a_{n,j}q^j$, $B_n(q) = \sum_{j=0}^{n^2-1} b_{n,j}q^j$ and $C_n(q) = \sum_{j=0}^{n^2-1} c_{n,j}q^j$, have the property

$$a_{n,i} = a_{n-1,i}, \quad b_{n,i} = b_{n-1,i}, \quad c_{n,i} = c_{n-1,i} \quad (i = 0, \ldots, n-2).$$

Moreover $a_{n,n-1} = a_{n-1,n-1}$.

According to Figure 1.b. we can believe that the next is true.

Conjecture 1.2. *The coefficients $a_{n,i}$ are increasing functions of index n, i.e.,*

$$a_{n,i} \le a_{n+1,i} \quad (i = 1, 2, \dots, [n^2/2]) \quad (n = 1, 2, \dots).$$

Starting from $n = 7$, the sequence $\{a_{n,i}\}$ becomes unimodal, i.e.

$$a_{n,i-1} \le a_{n,i} \quad (i = 1, 2, \dots, [n^2/2]) \qquad (n = 7, 8, \dots),$$

The Andrews' recurrence relations can be written in the next matrix form

$$X_n(q) = F(n, q) X_{n-1} \quad (n = 1, 2, \dots),$$

where

$$X_n(q) = \begin{bmatrix} A_n(q) \\ B_n(q) \\ C_n(q) \end{bmatrix}, \quad F(n, q) = \begin{bmatrix} (1 + q^{2n-1}) & q^n & q^n \\ q^{n-1} & (1 + q^{2n-1}) & -q^n \\ q^{n-1} & -q^{n-1} & (1 + q^{2n-1}) \end{bmatrix}.$$

It is can be of interest that

$$\det F(n, q) = 1 + (q^3)^{2n-1} - q(q^3)^{n-1} - q^2 (q^3)^{n-1}$$

has the similar expanding as $J(n; q)$.

Lemma 1.1. *The matrices $\{F(n, q)\}$ are commutative, i.e.*

$$F[k, q] F[n, q] = F[n, q] F[k, q].$$

Lemma 1.2. *The recurrence relations can be written in the form*

$$\begin{bmatrix} A_n & qC_n & qB_n \\ B_n & A_n & -qC_n \\ C_n & -B_n & A_n \end{bmatrix} = \begin{bmatrix} (1 + q^{2n-1}) & q^n & q^n \\ q^{n-1} & (1 + q^{2n-1}) & -q^n \\ q^{n-1} & -q^{n-1} & (1 + q^{2n-1}) \end{bmatrix} \begin{bmatrix} A_{n-1} & qC_{n-1} & qB_{n-1} \\ B_{n-1} & A_{n-1} & -qC_{n-1} \\ C_{n-1} & -B_{n-1} & A_{n-1} \end{bmatrix}.$$

Lemma 1.3. *The matrix $F(n,q)$ can be decomposed like*

$$F(n,q) = q^{n-1}L + (1+q^{2n-1})I + q^n L^t, \qquad \text{where} \qquad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

and L^t is transpose matrix of L.

Obviously,

$$L^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad (L^t)^2 = (L^2)^t \qquad L^n = (L^t)^n = 0 \quad (n = 3, 4\ldots)$$

By the mathematical induction, we can prove the next lemma.

Lemma 1.4. *It is valid*

$$(LL^t)^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f_{2n-2} & f_{2n-1} \\ 0 & f_{2n-1} & f_{2n} \end{bmatrix}, \quad (L^tL)^n = \begin{bmatrix} f_{2n} & -f_{2n-1} & 0 \\ -f_{2n-1} & f_{2n-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (n > 0),$$

where f_n are Fibonacci numbers:

$$f_0 = f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad (n = 2, 3, \ldots).$$

2. Borwein conjecture through eigenvalues

The matrix $F(n,q)$ is of great importance in Borwein conjecture. So, let us examine it by spectral analysis.

Theorem 2.1. *The matrix $F(n,q)$ has the eigenvalues*

$$\lambda_1(n;q) = (1-q^{n-2/3})(1-q^{n-1/3})$$

$$\lambda_2(n;q) = (1+\frac{1}{2}q^{n-2/3}+\frac{1}{2}q^{n-1/3}+q^{2n-1}) + \mathrm{i}\frac{\sqrt{3}}{2}q^{n-2/3}(1-q^{1/3})$$

$$\lambda_3(n;q) = (1+\frac{1}{2}q^{n-2/3}+\frac{1}{2}q^{n-1/3}+q^{2n-1}) - \mathrm{i}\frac{\sqrt{3}}{2}q^{n-2/3}(1-q^{1/3})$$

.

These eigenvalues have the next properties:

$$J(n;q) = \prod_{k=1}^{n} \lambda_1(k;q^3),$$

$$\lambda_1(n;q^3) = \lambda_1(n;q)\cdot\lambda_2(n;q)\cdot\lambda_3(n;q),$$

$$\lambda_2(n;q)\cdot\lambda_3(n;q) = (1+q^{n-1/3}+q^{2n-2/3})(1+q^{n-2/3}+q^{2n-4/3}).$$

Theorem 2.2. *The matrix $F(n,q)$ can be decomposed by*

$$F[n,q] = M \cdot D_n \cdot M^{-1},$$

where the matrix $D_n = \operatorname{diag}\{\lambda_1(n;q), \lambda_2(n;q), \lambda_3(n;q)\}$ and the matrices M and M^{-1} do not depend on index n and they are given by

$$M = \begin{bmatrix} -q^{2/3} & \frac{1-\mathrm{i}\sqrt{3}}{2}q^{2/3} & \frac{1+\mathrm{i}\sqrt{3}}{2}q^{2/3} \\ q^{1/3} & -\frac{1+\mathrm{i}\sqrt{3}}{2}q^{1/3} & \frac{-1+\mathrm{i}\sqrt{3}}{2}q^{1/3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{3q^{2/3}} \begin{bmatrix} -1 & q^{1/3} & q^{2/3} \\ \frac{1+\mathrm{i}\sqrt{3}}{2} & \frac{1-\mathrm{i}\sqrt{3}}{2}q^{1/3} & q^{2/3} \\ \frac{1-\mathrm{i}\sqrt{3}}{2} & -\frac{1+\mathrm{i}\sqrt{3}}{2}q^{1/3} & q^{2/3} \end{bmatrix}.$$

Theorem 2.3. *It is valid*

$$\prod_{i=1}^{n} F[i,q] = M \cdot \Big(\prod_{i=1}^{n} D_i\Big) \cdot M^{-1}.$$

Theorem 2.4. *The polynomial $A_n(q)$ can be expressed by*

$$A_n(q) = \frac{1}{3}\left\{\prod_{k=1}^{n} \lambda_1(k;q) + \prod_{k=1}^{n} \lambda_2(k;q) + \prod_{k=1}^{n} \lambda_3(k;q)\right\}.$$

3. Some reciprocal polynomials in the conjecture

We remind that a polynomial $A(q) = a_n q^n + a_{n-1}q^{n-1} + \cdots + a_0$ is *reciprocal* if $a_{n-k} = a_k$ $(k = 0, 1, \ldots, n)$, i.e., $A(q) = q^n A(1/q)$.

Yet G.E. Andrews [2] has noticed that

$$C_n(q) = q^{n^2-1} B_n(1/q) \quad (n \in \mathbb{N}), \tag{3.1}$$

i.e. that $B_n(q)$ and $C_n(q)$ are inversive to each other. Hence

$$c_{n,i} = b_{n,n^2-1-i} \quad (i = 0, 1 \ldots, n^2 - 1).$$

We can notice some sequences of reciprocal polynomials.

Let us denote by

$$D_n(q) = B_n(q) + C_n(q), \qquad E_n(q) = B_n(q) + qC_n(q).$$

The polynomials $A_n(q)$, $D_n(q)$ and $E_n(q)$ reciprocal, i.e.,

$$A_n(q) = q^{n^2} A_n(1/q), \quad D_n(q) = q^{n^2-1} D_n(1/q), \quad E_n(q) = q^{n^2-1} E_n(1/q).$$

The sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{E_n(q)\}$ satisfy the next recurrence relations

$$\begin{bmatrix} A_n \\ D_n \\ E_n \end{bmatrix} = \begin{bmatrix} 1+q^{2n-1} & q^n & 0 \\ 2q^{n-1} & 1+q^{2n-1} & -q^{n-1} \\ q^{n-1}+q^n & -q^n & 1+q^{2n-1} \end{bmatrix} \begin{bmatrix} A_{n-1} \\ D_{n-1} \\ E_{n-1} \end{bmatrix}, \qquad \begin{bmatrix} A_0 \\ D_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly, introducing reciprocal polynomial

$$F_n = A_n + E_n,$$

we can prove

The sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{F_n(q)\}$ satisfy the next recurrence relations

$$\begin{bmatrix} A_n \\ D_n \\ F_n \end{bmatrix} = \begin{bmatrix} 1+q^{2n-1} & q^n & 0 \\ 3q^{n-1} & 1+q^{2n-1} & -q^{n-1} \\ q^{n-1}+q^n & 0 & 1+q^{2n-1} \end{bmatrix} \begin{bmatrix} A_{n-1} \\ D_{n-1} \\ F_{n-1} \end{bmatrix}, \qquad \begin{bmatrix} A_0 \\ D_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

4. The fundamental recurrence relation

It seems to be of great importance to find separate recurrence relations for the sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$.

Theorem 4.1. *The sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$ satisfy the same recurrence relation*

$$\begin{aligned} \mathcal{F}_{n+3} = & \\ & \mathcal{F}_{n+2} \cdot (1+q+q^2)(1+q^{2n+3}) \\ - & \mathcal{F}_{n+1} \cdot q(1+q+q^2)(1+q^{2n+2}+q^{4n+4}) \\ + & \mathcal{F}_n \cdot q^3(1-q^{3n+1})(1-q^{3n+2}), \end{aligned}$$

with only difference in initial values:

$$\begin{aligned}
\mathcal{F}_n = A_n(q):&\quad A_0 = 1, \quad A_1 = 1+q, \quad A_2(q) = 1+q+2q^2+q^3+q^4,\\
\mathcal{F}_n = B_n(q):&\quad B_0 = 0, \quad B_1 = 1, \quad B_2(q) = 1+q+q^3,\\
\mathcal{F}_n = C_n(q):&\quad C_0 = 0, \quad C_1 = 1, \quad C_2(q) = 1+q^2+q^3,\\
\mathcal{F}_n = D_n(q):&\quad D_0 = 0, \quad D_1 = 2, \quad D_2(q) = 2+q+q^2+2q^3,\\
\mathcal{F}_n = E_n(q):&\quad E_0 = 0, \quad E_1 = 1+q, \quad E_2(q) = 1+2q+2q^3+q^4,\\
\mathcal{F}_n = F_n(q):&\quad F_0 = 1, \quad F_1 = 2+2q, \quad F_2(q) = 2+3q+2q^2+3q^3+2q^4.
\end{aligned}$$

Proof. The most simple way to find the recurrence relation is from the relations for the sequences $\{A_n(q)\}$, $\{D_n(q)\}$ and $\{F_n(q)\}$ which are given in Theorem 3.3. From the second relation we have

$$q^{n-1}F_{n-1} = 3q^{n-1}A_{n-1} + (1+q^{2n-1})D_{n-1} - D_n \quad (n \in \mathbb{N}).$$

Now, we can eliminate F_{n-1} and F_n from the third relation, i.e. we have

$$\begin{aligned}
3q^n A_n - (q^{2n-1}(1+q) + 3q^n(1+q^{2n-1}))A_{n-1}&\\
= D_{n+1} - (1+q)(1+q^{2n})D_{n-1} &+ q(1+q^{2n-1})^2 D_{n-1}.
\end{aligned}$$

From the first relation of Theorem 3.3, we have

$$q^n D_{n-1} = A_n - (1+q^{2n-1})A_{n-1} \quad (n \in \mathbb{N}).$$

We use it for D_{n-1}, D_n and D_{n+1} and put in the previous relation. After changing $n \to n+1$, we get the wanted difference equation. □

The recurrence relation for $A_n(q)$ can be written in the form

$$\begin{aligned}
A_{n+3}(q) = (1+q+q^2)&\left\{(1+q^{2n+3})A_{n+2}(q) - q(1+q^{2n+2})^2 A_{n+1}(q)\right\}\\
&+ q^3\left\{q^{2n}(1+q+q^2)A_{n+1}(q) + (1-q^{3n+1})(1-q^{3n+2})A_n(q)\right\}.
\end{aligned}$$

Conjecture 4.2. *The polynomials*

$$(1+q^{2n+3})A_{n+2}(q) - q(1+q^{2n+2})^2 A_{n+1}(q)$$

and

$$q^{2n}(1+q+q^2)A_{n+1}(q) + (1-q^{3n+1})(1-q^{3n+2})A_n(q)$$

are the polynomials with non-negative coefficients.

5. The zeros

Let us remind that a polynomial

$$P(q) = q^k - 1 \quad (k \in \mathbb{N})$$

has the zeros

$$q_{k,j} = \exp(i\frac{2\pi}{k}(j-1)) \quad (j = 1, 2, \dots, k)$$

which have the property

$$\sum_{j=1}^{k} q_{k,j}^m = k\ \delta_{\mathrm{mod}(m,k),0} = \begin{cases} k, & \mathrm{mod}(m,k) = 0 \\ 0, & \mathrm{mod}(m,k) \neq 0. \end{cases}$$

We will denote by s_m the sum of m-the powers of all zeros of $J(n,q)$, i.e.

$$s_m = \sum_{k=0}^{n-1} \left(\sum_{j=1}^{3k+1} q_{3k+1,j}^m + \sum_{j=1}^{3k+2} q_{3k+2,j}^m \right).$$

Hence

$$s_m = \sum_{k=0}^{N} \left\{ (3k+1)\ \delta_{k',0}\ +\ (3k+2)\ \delta_{k",0} \right\}, \tag{5.1}$$

where $\delta_{k,j}$ is Kronecker delta and

$$N = \min\{[m/3], n-1\}, \quad k' = \mathrm{mod}(m, 3k+1);\ k" = \mathrm{mod}(m, 3k+2). \tag{5.2}$$

Generally,

$$1 \leq s_m \leq n(3n+1)/2 \qquad (m = 1, 2, \dots).$$

If m is a prime number greater than $3n-1$, than $s_m = 1$. Also, $s_{3^k} = 1$ ($k = 1, 2, \dots$).

Lemma 5.1. *It is valid*

$$s_{3m} = s_m \qquad (m > 3n).$$

Proof. Obviously, if $\mathrm{mod}(m, 3k+1) = 0$, then $\mathrm{mod}(3m, 3k+1) = 0$, and if $\mathrm{mod}(m, 3k+1) \neq 0$, then $\mathrm{mod}(3m, 3k+1) \neq 0$. Similar conclusion is valid if we change $3k+1$ by $3k+2$. Hence

$$\delta_{\mathrm{mod}(3m,3k+1),0} = \delta_{\mathrm{mod}(m,3k+1),0}, \qquad \delta_{\mathrm{mod}(3m,3k+2),0} = \delta_{\mathrm{mod}(m,3k+2),0}.$$

At last, for any $m > 3n$, in the formula $(5.1-2)$, the number N is equal $n-1$. $\square$

Now, the coefficients of the polynomial

$$J(n,q) = q^{3n^2} + d_{n,1}q^{3n^2-1} + \cdots + d_{n,3n^2-1}q + d_{n,3n^2}$$

and the sums s_m are connected by Newton's formulae

$$s_m + d_{n,1}s_{m-1} + \cdots + d_{n,m-1}s_1 + md_{n,m} = 0 \qquad (m = 1, 2, \ldots, 3n^2).$$

wherefrom we get much faster algorithm for evaluating of the coefficients $d_{n,k}$.

Conjecture 5.1. *The sums $\{s_m\}$ and coefficients $\{d_{n,i}\}$ satisfy the next relation*

$$s_{3i-2} + s_{3i-1} \geq s_{3i} \leq s_{3i+1} + s_{3i+2}$$

$$|d_{n,3i-2}| + |d_{n,3i-1}| \leq d_{n,3i} \leq |d_{n,3i+1}| + |d_{n,3i+2}|$$

$$(i = 1, 2, \ldots, [n^2]/2) \quad (n = 1, 2, \ldots).$$

It can be of interest to examine the zeros of polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$.

Lemma 5.1. *The polynomial $A_{2n+1}(q)$ vanishes in the point -1. Moreover, it can be written in the form*

$$A_{2n+1}(q) = (1+q)\hat{A}_{2n+1}(q),$$

where $\hat{A}_{2n+1}(q)$ is a reciprocal polynomial.

Proof. From recurrence relation for the sequence $\{A_n(q)\}$, taking $q = -1$, we obtain $A_{2n+3}(-1)= 3A_{2n+1}(-1)$. How it is $A_1(-1) = 0$, by mathematical induction, we have $A_{2n+1}(-1) = 0$ for all $n \in \mathbb{N}$. Hence, we can write $A_{2n+1}(q) = (1+q)\hat{A}_{2n+1}(q)$, where $\hat{A}_{2n+1}(q)$ is a polynomial of degree $(2n+1)^2-1$. Knowing that $A_{2n+1}(q)$ is the reciprocal polynomial,i.e. $A_{2n+1}(q) = q^{(2n+1)^2}A_{2n+1}(1/q)$, by simple change, we have

$$\hat{A}_{2n+1}(q) = q^{(2n+1)^2-1}\hat{A}_{2n+1}(1/q).\square$$

Conjecture 5.2. *The polynomial $\hat{A}_{2n+1}(q)$ has all positive coefficients.*

Lemma 5.2. *If the polynomial $A_n(q)$ vanishes in the point $z = Re^{\mathrm{it}}$ then it also vanishes at the points $z = Re^{-\mathrm{it}}$, $z = e^{\mathrm{it}}/R$ and $z = e^{-\mathrm{it}}/R$.*

Proof. How $A_n(q)$ has all real coefficients, it follows that the complex zeros appear in conjugate pairs. From the fact that $A_n(q)$ is the reciprocal polynomial we conclude that, if $z = Re^{\mathrm{it}}$ is a zero, then also is $1/z = e^{-\mathrm{it}}/R$. □

Our numerical evaluating persuade us that all zeros of $A_n(q)$ lie in the ring $1 - \varepsilon < |z| < 1 + \varepsilon$, where $0 < \varepsilon < 1/2$.

Especially, when we consider a zero $z = Re^{\mathrm{it}}$ ($R \neq 0$), according to Lemma 8.2, we have 4 different zeros and the next product

$$(z - Re^{\mathrm{it}})(z - Re^{-\mathrm{it}})(z - e^{\mathrm{it}}/R)(z - e^{-\mathrm{it}}/R)$$
$$= z^4 - 2\big(R + \frac{1}{R}\big) \cos t \ \cdot z(1 + z^2) + \left\{ \big(R + \frac{1}{R}\big)^2 + 2\cos(2t) \right\} z^2 + 1$$

in the polynomial $A_n(q)$.

In the case $R = 1$, this product can be written like $(z^2 - 2z\cos t + 1)^2$. But, only one or two zeros appear from this four point set.

Similar behavior we notice for the zeros of $B_n(q)$ and $C_n(q)$.

6. No doubt, the positive sequences

The sequences of polynomials $\{A_n^+(q)\}$, $\{B_n^+(q)\}$ and $\{C_n^+(q)\}$ derived from the relation

$$\prod_{i=1}^{n}(1 + q^{3i-2})(1 + q^{3i-1}) = A_n^+(q^3) + qB_n^+(q^3) + q^2C_n^+(q^3),$$

are, no doubt, with positive coefficients.

Remembering the Borwein conjecture

$$\prod_{i=1}^{n}(1 - q^{3i-2})(1 - q^{3i-1}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3),$$

and by multiplying the same sides of equalities, we can prove the next theorem.

Theorem 6.1. *The sequences* $\{A_n(q)\}$, $\{B_n(q)\}$ *and* $\{C_n(q)\}$ *are connected with the sequences* $\{A_n^+(q)\}$, $\{B_n^+(q)\}$ *and* $\{C_n^+(q)\}$, *by the next relations*

$$\begin{aligned}A_n(q^2) &= A_n^+(q)A_n(q) - qC_n^+(q)B_n(q) - qB_n^+(q)C_n(q),\\ B_n(q^2) &= -C_n^+(q)A_n(q) + B_n^+(q)B_n(q) + A_n^+(q)C_n(q)\\ qC_n(q^2) &= -B_n^+(q)A_n(q) + A_n^+(q)B_n(q) + qC_n^+(q)C_n(q).\end{aligned}$$

7. The generating functions and transforms

Denote by

$$\alpha(z) = \sum_{n=0}^{\infty} A_n(q)z^n, \quad \beta(z) = \sum_{n=0}^{\infty} B_n(q)z^n, \quad \gamma(z) = \sum_{n=0}^{\infty} C_n(q)z^n$$

the generating functions of the sequences of polynomials included in Borwein conjecture.

Theorem 7.1. *The generating functions* $\alpha(z)$, $\beta(z)$ *and* $\gamma(z)$ *satisfy the next system of the functional equations*

$$\begin{aligned}z^{-1}\big[\alpha(z)-1\big] &= \alpha(z) + q\alpha(q^2z) + q\beta(qz) + q\gamma(qz),\\ z^{-1}\beta(z) &= \alpha(qz) + \beta(z) + q\beta(q^2z) - q\gamma(qz),\\ z^{-1}\gamma(z) &= \alpha(qz) - \beta(qz) + \gamma(z) + q\gamma(q^2z),\end{aligned}$$

with initial values

$$\alpha(0) = A_0 = 1, \quad \beta(0) = B_0 = 0, \quad \gamma(0) = C_0 = 0.$$

Using the fundamental recurrence relation, we prove

Theorem 7.2. *The generating function* $\alpha(z)$ *satisfies the next functional equation*

$$\begin{aligned}&q^7z^3\ \alpha(q^6z) - (1+q+q^2)\big\{(qz)^2\ \alpha(q^4z) + q^4z^3\ \alpha(q^3z) - (1-qz)z\ \alpha(q^2z)\big\}\\ &-q(1-z)(1-qz)(1-q^2z)\ \alpha(z) + q\big[(1-q-2q^2-q^3+q^4+q^5)z + 1 + 2(1+q+q^2)\big] = 0.\end{aligned}$$

Let us apply Laplace transform on the sequences $\{A_n(q)\}$, $\{B_n(q)\}$ and $\{C_n(q)\}$. Denote by

$$\mathcal{L}\big[A_n(q)\big] = a_n(p), \quad \mathcal{L}\big[B_n(q)\big] = b_n(p), \quad \mathcal{L}\big[C_n(q)\big] = c_n(p).$$

Knowing that

$$\mathcal{L}\big[q^k\big] = \frac{k!}{q^{k+1}}, \qquad \mathcal{L}\big[q^kA_n(q)\big] = (-1)^k a_n^{(k)}(p),$$

we have

Theorem 7.3. *The Borwein conjecture can be written in the form*

$$
\begin{aligned}
a_n(p) &= a_{n-1}(p) - a_{n-1}^{(2n-1)}(p) + (-1)^n b_{n-1}^{(n)}(p) + (-1)^n c_{n-1}^{(n)}(p),\\
b_n(p) &= (-1)^{n-1} a_{n-1}^{(n-1)}(p) + b_{n-1}(p) - b_{n-1}^{(2n-1)}(p) - (-1)^n c_{n-1}^{(n)}(p),\\
c_n(p) &= (-1)^{n-1} a_{n-1}^{(n-1)}(p) - (-1)^{n-1} b_{n-1}^{(n-1)}(p) + c_{n-1}(p) - c_{n-1}^{(2n-1)}(p).
\end{aligned}
$$

Here, the improvement is in the fact that we find recurrence relations with constant coefficients.

Also, we can apply the transform $\mathbf{q\ D\ ln}\ \cdot$, where $\mathbf{D}$ is derivative by q. We yield

$$
\mathbf{q\ D\ ln}\ J_n(q) = \frac{3q^3 A_n'(q^3) - q\big[B_n(q^3) + 3q^3 B_n'(q^3)\big] - q^2\big[2C_n(q^3) + 3q^3 C_n'(q^3)\big]}{A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3)},
$$

i.e., numerator and denominator hold on the property $+--$.

8. Some expansions

Introducing two new variables $x = q$ and $t = q^3$ into $J(n;q)$, we can consider a new function $f_n(x,t)$ defined by

$$
f_n(x,t) = \prod_{i=0}^{n-1} (1 - xt^i)(1 - x^{-1}t^{i+1}) \qquad (0 < x, t < 1). \tag{8.1}
$$

Expanding (8.1) over powers of x, we have

$$
f_n(x,t) = \sum_{k=-n}^{n} c_{n,k} x^k.
$$

Lemma 8.1. *The function $f(x,t)$ has the next properties*

$$
f_n(tx,t) = f_n(x^{-1},t) = \frac{xt^n - 1}{x - t^n} f_n(x,t), \quad f_n(x/t,t) = \frac{t^{n+1} - x}{t(1 - xt^{n-1})} f_n(x,t).
$$

Lemma 8.2. *The coefficients $c_{n,k}$ satisfy the next relations:*

$$\begin{aligned} c_{n,-j} &= t^j c_{n,j} \\ c_{n,j} &= -t^{j-1}\frac{t^{n-j+1}-1}{t^{n+j}-1}c_{n,j-1} \\ &\quad (j=-n+1,-n+2,\dots,n) \end{aligned}$$

with boundary values

$$c_{n,-n} = (-1)^n t^{n(n+1)/2} \qquad c_{n,n} = (-1)^n t^{(n-1)n/2}.$$

Lemma 8.3. *Polynomials $A_n(t)$, $B_n(t)$ and $C_n(t)$ can be written in the form*

$$\begin{aligned} A_n(t) &= c_{n,0} + \sum_{k=1}^{[\frac{n}{3}]} c_{n,3k}t^k(t^k+1) \\ B_n(t) &= -\sum_{k=1}^{[\frac{n-1}{3}]} c_{n,3k+1}t^k(t^k+1) \\ C_n(t) &= -\sum_{k=1}^{[\frac{n-2}{3}]} c_{n,3k+2}t^k(t^k+1) \end{aligned}$$

Denote $G(n,j,t) = -t^{j-1}\frac{t^{n-j+1}-1}{t^{n+j}-1} = -\frac{t^n-t^{j-1}}{t^{n+j}-1}$. Rewrite $c_{n,j}$ in form $c_{n,j}(q) = \sum c_{n,j,k}q^k$.

Lemma 8.4. *Coefficients $c_{n,j,k}$ satisfies next recurrence relation:*

$$c_{n,j,k} = \sum_{p=0}^{[\frac{k-n}{n+j}]} c_{n,j-1,k-n-p(n+j)} - \sum_{p=0}^{[\frac{k-j+1}{n+j}]} c_{n,j-1,k-j+1-p(n+j)}$$

Proof.

Let us start from $c_{n,j} = G(n,j,q)c_{n,j-1}$ and $c_{n,j,k} = \frac{c^{(k)}_{n,j}(0)}{k!}$. Derivative $c^{(k)}_{n,j}(q)$ we will compute using Leibnitz formula:

$$c^{(k)}_{n,j}(0) = \sum_{i=0}^{k}\binom{k}{i}G^{(i)}(n,j,0)c^{(k-i)}_{n,j-1}(0)$$

For derivatives $G^{(i)}(n,j,t)$ we will also apply Leibnitz formula:

$$G^{(i)}(n,j,t)=-\sum_{s=0}^{i}\binom{i}{s}\left(t^n-t^{j-1}\right)^{(s)}\left(\frac{1}{t^{n+j}-1}\right)^{(i-s)}$$

For $t=0$ expression $\left(t^n-t^{j-1}\right)^{(s)}$ vanishes for $s\neq j-1,n$. So upper sum can be rewritten in the form:

$$G^{(i)}(n,j,0)=\binom{i}{j-1}(j-1)!\left(\frac{1}{t^{n+j}-1}\right)^{(i-j+1)}(0)-\binom{i}{n}n!\left(\frac{1}{t^{n+j}-1}\right)^{(i-n)}(0)$$

It can be proven that holds $\left(\frac{1}{t^{n+j}-1}\right)^{(s)}(0)=-s!$ for $n+j|s$ and otherwise it is 0.Using this we can calculate derivative $G^{(i)}(n,j,t)$:

$$G^{(i)}(n,j,0)=i!\cdot\sum_{p=0}^{i}\left(\Delta_{p(n+j),i-j+1}-\Delta_{p(n+j),i-n}\right)$$

Again, substitution in equation () completes the proof.

Letting n to tends to infinity we get

$$f_\infty(x,t)=\prod_{i=0}^{\infty}(1-xt^i)(1-x^{-1}t^{i+1})\qquad(0<x,t<1),$$

and the properties

$$f_\infty(tx,t)=f_\infty(x^{-1},t)=\frac{-1}{x}f_\infty(x,t),\qquad f_\infty(x/t,t)=\frac{-x}{t}f_\infty(x,t).$$

Writing $f_\infty(x,t)$ in the form

$$f_\infty(x,t)=\prod_{k=1}^{\infty}\frac{1-x^k}{1-t^k}\qquad(0<x,t<1),$$

we conclude

$$f_\infty(t,x)=f_\infty(x,t).$$

At the end of this section we will give some conjectures.

Conjecture 8.1. *Polynomials*

$$c_{n,3k}(q)\cdot q^k(q^k+1)+c_{n,3(k+1)}(q)\cdot q^{k+1}(q^{k+1}+1)$$

has positive coefficients for all even k*,* $2\le k\le [n/3]$*.*

Conjecture 8.2. *Polynomial*

$$\sum_{k=0}^{3} c_{n,3k}(q)\cdot q^k(q^k+1)$$

has positive coefficients.

Conjecture 8.3. *Polynomials* $c_{n,j}(q)$ *have positive coefficients for even and negative for odd* j*.*

9. Conjectures for further research

By using package *Mathematica*, we have done a lot of trials which persuade us that the next statements are true.

Conjecture 9.1. *The polynomial* $B_n(q)=\sum_{j=0}^{n^2-1} b_{n,j}q^j$ *has the property*

$$b_{n,n^2-2}=0,\quad b_{n,j}>0,\ j\neq n^2-2.$$

Equivalently, knowing $C_n(q)=q^{n^2-1}B_n(1/q)\quad (n\in\mathbb{N})$, we can establish the next conjecture.

Conjecture 9.2. *The polynomial* $C_n(q)=\sum_{j=0}^{n^2-1} c_{n,j}q^j$ *has the property*

$$c_{n,1}=0,\quad c_{n,j}>0,\ j\neq 1.$$

Conjecture 9.3. *The coefficients of the polynomials* $A_n(q)-B_n(q)$ *and* $A_n(q)-qB_n(q)$ *are positive.*

Conjecture 9.4. *The coefficients of the polynomial* $A_n(q)-C_n(q)$ *and* $A_n(q)-qC_n(q)$ *are positive.*

Conjecture 9.5. *The coefficients of the polynomials* $A_n(q) - (1+q^{2n-1})A_{n-1}(q)$, $B_n(q) - (1+q^{2n-1})B_{n-1}(q)$ *and* $C_n(q) - (1+q^{2n-1})C_{n-1}(q)$ *are positive.*

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