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Research Article

Functions Induced by Iterated Deformed Laguerre Derivative: Analytical and Operational Approach

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The one-parameter deformed exponential function was introduced as a frame that enchases a few known functions of this type. Such deformation requires the corresponding deformed operations (addition and subtraction) and deformed operators (derivative and antiderivative). In this paper, we will demonstrate this theory in researching of some functions defined by iterated deformed Laguerre operator. We study their properties, such as representation, orthogonality, generating function, differential and difference equation, and addition and summation formulas. Also, we consider these functions by the operational method.

1. Introduction

The several parametric generalizations and deformations of the exponential function have been proposed recently in different contexts such as nonextensive statistical mechanics [1, 2], relativistic statistical mechanics [3, 4], and quantum group theory [5–7].

The areas of deformations of the exponential functions have been treated basically along three (complementary) directions: formal mathematical developments, observation of consistent concordance with experimental (or natural) behavior, and theoretical physical developments.

In paper [8], a deformed exponential function of two variables depending on a real parameter is introduced to express discrete and continual behavior by the same. In this function, well-known generalizations and deformations can be viewed as the special cases [1, 5]. Also, its usage can be seen in [9].

In continuation of our previous considerations of the deformed exponential function, we will use its convenience to introduce and research a class of functions of two variables that can be viewed as one-parameter analog of Laguerre polynomials.

The paper is organized as follows. In Section 2, we introduce the deformed exponential function and the related deformation of variable, addition and subtraction. In Section 3, we consider some known and new difference and differential operators, convenient for the work with deformed variables and exponentials. In this environment, in Section 4, we define a class of two-variable functions, called the deformed Laguerre polynomials, by an iterated generalized differential operator. Section 5 is devoted to various properties of these functions, such as orthogonality, summation and addition formulas, differential equations, generating function. Finally, in Section 6 we prove a few operational identities involving the introduced deformed Laguerre polynomials.

Because of the presence of "logarithmic scale," deformed Laguerre operator and deformed Laguerre polynomials could be suitable for use in control engineering, population dynamics, mathematical modeling of viscous fluids, and oscillating problems in mechanics, like the usual Laguerre operator and Laguerre polynomials that are already used (see [10–12]).

2. The Deformed Exponential Functions

In this section we will present a deformation of an exponential function of two variables depending on parameter $h \in \mathbb{R} \setminus \{0\}$, which is introduced in [8].

Let us define function $(x, y) \mapsto e_h(x, y)$ by

$$e_h(x,y) = (1+hx)^{y/h} \quad \left(x \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\},\ y \in \mathbb{R}\right). \tag{2.1}$$

Since

$$\lim_{h\to 0} e_h(x,y) = e^{xy}, \tag{2.2}$$

this function can be viewed as a one-parameter deformation of the exponential function of two variables.

If $h = 1 - q$ $(q \neq 1)$ and $y = 1$, function (2.1) becomes

$$e_{1-q}(x,1) = \left(1+(1-q)x\right)^{1/(1-q)}, \tag{2.3}$$

that is, $e_{1-q}(x,1) = e_q^x$, where e_q^x is Tsallis q-exponential function [1] defined by

$$e_q^x = \begin{cases} \left(1+(1-q)x\right)^{1/(1-q)}, & 1+(1-q)x > 0 \\ 0, & \text{otherwise} \end{cases} \quad (x \in \mathbb{R}). \tag{2.4}$$

If $h = p - 1$ $(p \neq 1)$ and $x = 1$, function (2.1) becomes

$$e_{p-1}(1,y) = p^{y/(p-1)}, \tag{2.5}$$

that is, a function considered for a generalization of the standard exponential function in the context of quantum group formalism [13].

Notice that function (2.1) can be written in the form

$$e_h(x,y) = \exp\left(\frac{y}{h}\ln(1+hx)\right). \tag{2.6}$$

Hence, similar to what in [14], we can use cylinder transformation as deformation function $x \mapsto \{x\}_h$ by

$$\{x\}_h = \frac{1}{h}\ln(1+hx) = \ln(1+hx)^{1/h} \quad \left(x \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\}\right). \tag{2.7}$$

Thus, the following holds:

$$e_h(x,y) = e^{\{x\}_h\, y}. \tag{2.8}$$

We can show that function (2.1) holds on some basic properties of the exponential function.

Proposition 2.1. *For $x \in \mathbb{C} \setminus \{-1/h\}$ and $y, y_1, y_2 \in \mathbb{R}$, the following holds:*

$$\begin{gathered} e_h(x,y) > 0 \quad \left(x < -\frac{1}{h} \text{ for } h < 0 \text{ or } x > -\frac{1}{h} \text{ for } h > 0\right), \\ e_h(0,y) = e_h(x,0) = 1, \\ e_{-h}(x,y) = e_h(-x,-y) \quad \left(x \neq \frac{1}{h}\right), \\ e_h(x, y_1 + y_2) = e_h(x,y_1)e_h(x,y_2). \end{gathered} \tag{2.9}$$

Notice that the additional property is true with respect to the second variable only. However, with respect to the first variable, the following holds:

$$e_h(x_1,y)e_h(x_2,y) = e_h(x_1 + x_2 + hx_1x_2, y). \tag{2.10}$$

This equality suggests introducing a generalization of the sum operation

$$x_1 \oplus_h x_2 = x_1 + x_2 + hx_1x_2. \tag{2.11}$$

Such generalized addition operator was considered in some papers and books (see, e.g., [2] or [14]). This operation is commutative and associative, and zero is its neutral. For $x \neq -1/h$, the $\ominus_h$-inverse exists as

$$\ominus_h x = \frac{-x}{1+hx}, \tag{2.12}$$

and $x \oplus_h (\ominus_h x) = 0$ is valid. Hence, $(I, \oplus_h)$ is an abelian group, where $I = (-\infty, -1/h)$ for $h < 0$ or $I = (-1/h, +\infty)$ for $h > 0$. In this way, the $\ominus_h$-subtraction can be defined by

$$x_1 \ominus_h x_2 = x_1 \oplus_h (\ominus_h x_2) = \frac{x_1 - x_2}{1 + hx_2} \quad \left(x_2 \neq -\frac{1}{h} \right). \tag{2.13}$$

With respect to (2.7), we can prove the next equality for $x_1, x_2 \in I$:

$$\{x_1\}_h + \{x_2\}_h = \{x_1 \oplus_h x_2\}_h. \tag{2.14}$$

Proposition 2.2. *For $x_1, x_2 \in \mathbb{C} \setminus \{-1/h\}$ and $y \in \mathbb{R}$, the following is valid:*

$$\begin{aligned} e_h(x_1 \oplus_h x_2, y) &= e_h(x_1, y) e_h(x_2, y), \\ e_h(x_1 \ominus_h x_2, y) &= e_h(x_1, y) e_h(x_2, -y). \end{aligned} \tag{2.15}$$

In order to find the expansions of the introduced deformed exponential function, we introduce the generalized backward integer power given by

$$z^{(0,h)} = 1, \quad z^{(n,h)} = \prod_{k=0}^{n-1} (z - kh) \quad (n \in \mathbb{N},\ h \in \mathbb{R} \setminus \{0\}). \tag{2.16}$$

Proposition 2.3. *For function $(x, y) \mapsto e_h(x, y)$, the following representation holds:*

$$e_h(x, y) = \sum_{n=0}^{\infty} \frac{\{x\}_h^n y^n}{n!}, \quad e_h(x, y) = \sum_{n=0}^{\infty} \frac{x^n y^{(n,h)}}{n!} \quad (|hx| < 1). \tag{2.17}$$

Remark 2.4. Notice that in expressions (2.8) and the first expansion in (2.17) the deformation of variable x appears, but, contrary to that in the second expansion in (2.17), the deformation of powers of y is present.

3. The Deformed Operators

Let us recall that the h-difference operator is

$$\Delta_{z,h} f(z) = \frac{f(z + h) - f(z)}{h}. \tag{3.1}$$

Proposition 3.1 (see [8]). *The function $y \mapsto e_h(x, y)$ is the eigenfunction of difference operator $\Delta_{y,h}$ with eigenvalue x, that is, the following holds:*

$$\Delta_{y,h}\, e_h(x, y) = x\, e_h(x, y). \tag{3.2}$$

Also, there are a few differential operators that have deformed the exponential function as eigenfunction.

Let us define the deformed h-differential and h-derivative accordingly with operation (2.11) (see [15]):

$$d_h z = \lim_{u\to z} z\ominus_h u, \qquad D_{z,h} f(z) = \frac{df(z)}{d_h z} = \lim_{u\to z} \frac{f(z) - f(u)}{z\ominus_h u}. \tag{3.3}$$

With respect to (2.13), we have

$$D_{z,h} f(z) = \frac{df(z)}{d_h z} = \lim_{u\to z} \frac{f(z) - f(u)}{(z-u)/(1+hu)} = (1+hz)\frac{df(z)}{dz}. \tag{3.4}$$

The h-derivative holds on the property of linearity and the product rule:

$$\begin{aligned} D_{z,h}\big(\alpha f(z) + \beta g(z)\big) &= \alpha D_{z,h} f(z) + \beta D_{z,h} g(z), \\ D_{z,h}\big(f(z)g(z)\big) &= f(z) D_{z,h} g(z) + g(z) D_{z,h} f(z). \end{aligned} \tag{3.5}$$

Let $I = (-\infty, -1/h)$ for $h < 0$ or $I = (-1/h, +\infty)$ for $h > 0$. For $x \in I$, we define the inverse operator to operator $D_{x,h}$ (inverse up to a constant) by

$$D_{x,h}^{-1} f(x) = \int_0^x \frac{f(t)}{1+ht}\, dt. \tag{3.6}$$

It is easy to prove that

$$D_{x,h}^{-n} f(x) = \left(D_{x,h}^{-1}\right)^n f(x) = \frac{1}{(n-1)!}\int_0^x \frac{\left(\{x\}_h - \{t\}_h\right)^{n-1}}{1+ht} f(t)dt. \tag{3.7}$$

Proposition 3.2. *The function $e_h(x,y)$ is the eigenfunction of the operators $D_{x,h}$ and $\partial/\partial y$ with eigenvalues y and $\{x\}_h$, respectively, that is,*

$$D_{x,h} e_h(x,y) = y e_h(x,y), \qquad \frac{\partial}{\partial y} e_h(x,y) = \{x\}_h e_h(x,y). \tag{3.8}$$

Moreover, for $y \neq 0$, the following is valid:

$$D_{x,h}^{-1} e_h(x,y) = \frac{1}{y} e_h(x,y) - \frac{1}{y}. \tag{3.9}$$

Lemma 3.3. *For $x \in I$, $y \in \mathbb{R}$, and $k, n \in \mathbb{N}_0$, the following holds:*

$$D_{x,h}^{k}\left(\{x\}_h^n\right) = k!\binom{n}{k}\{x\}_h^{n-k}, \tag{3.10}$$

$$D_{x,h}^{-k}(1) = \frac{\{x\}_h^k}{k!}, \tag{3.11}$$

$$\left(1 - yD_{x,h}^{-1}\right)^{-1}(1) = e_h(x, y). \tag{3.12}$$

Proof. Equalities (3.10) and (3.11) follow from definition (2.7), (3.4), and (3.6). For (3.12), we recall the expansion (2.17) and the formal geometric series:

$$\begin{aligned} e_h(x, y) &= \sum_{n=0}^{\infty} \frac{\{x\}_h^n y^n}{n!} = \sum_{n=0}^{\infty} y^n D_{x,h}^{-n}(1) = \sum_{n=0}^{\infty} \left(yD_{x,h}^{-1}\right)^n (1) \\ &= \left(1 - yD_{x,h}^{-1}\right)^{-1}(1). \end{aligned} \tag{3.13}$$

□

Furthermore, let us introduce a multiplicative operator

$$X_h f(x) = \{x\}_h f(x). \tag{3.14}$$

Lemma 3.4. *For $n \in \mathbb{N}_0$, the following holds:*

$$D_{x,h} X_h^n - X^n D_{x,h} = nX_h^{n-1}, \qquad D_{x,h}^n X_h - X_h D_{x,h}^n = nD_{x,h}^{n-1}, \tag{3.15}$$

$$X_h D_{x,h}^{n+1} X_h^n = D_{x,h}^n X_h^{n+1} D_{x,h}. \tag{3.16}$$

Proof. Using the product rule for $D_{x,h}$, we get equalities (3.15) for $n = 1$:

$$(D_{x,h} X_h) f(x) = D_{x,h}(\{x\}_h f(x)) = f(x) + \{x\}_h D_{x,h} f(x) = (1 + X_h D_{x,h}) f(x). \tag{3.17}$$

Hence,

$$X_h D_{x,h}^2 X_h - D_{x,h} X_h^2 D_{x,h} = X_h D_{x,h}(X_h D_{x,h} + 1) - (X_h D_{x,h} + 1) X_h D_{x,h} = 0, \tag{3.18}$$

which proves equality (3.16) for $n = 1$. The cases for $n > 1$ can be proved by induction or by repeated procedure:

$$\begin{aligned} X_h D_{x,h}^{n+1} X_h^n - D_{x,h}^n X_h^{n+1} D_{x,h} &= X_h D_{x,h}^n (D_{x,h} X_h^n) - \left(D_{x,h}^n X_h\right) X_h^n D_{x,h} \\ &= X_h D_{x,h}^n \left(X_h^n D_{x,h} + nX_h^{n-1}\right) - \left(X_h D_{x,h}^n + nD_{x,h}^{n-1}\right) X_h^n D_{x,h} \\ &= n\left(X_h D_{x,h}^n X_h^{n-1} - D_{x,h}^{n-1} X_h^n D_{x,h}\right) = \cdots = 0. \end{aligned} \tag{3.19}$$

□

Theorem 3.5. *For $n \in \mathbb{N}_0$, the following is valid:*

$$(D_{x,h} X_h D_{x,h})^n = \left(D_{x,h}^n X_h^n D_{x,h}^n\right). \tag{3.20}$$

Proof. The statement is obviously true for $n = 1$. Suppose that formula is true for n. According to Lemma 3.4, we have

$$
\begin{aligned}
(D_{x,h}X_hD_{x,h})^{n+1} - D_{x,h}^{n+1}X_h^{n+1}D_{x,h}^{n+1} &= (D_{x,h}X_hD_{x,h})(D_{x,h}X_hD_{x,h})^n - D_{x,h}^{n+1}X_h^{n+1}D_{x,h}^{n+1} \\
&= (D_{x,h}X_hD_{x,h})\left(D_{x,h}^nX_h^nD_{x,h}^n\right) - D_{x,h}^{n+1}X_h^{n+1}D_{x,h}^{n+1} \\
&= D_{x,h}\left(X_hD_{x,h}^{n+1}X_h^n - D_{x,h}^nX_h^{n+1}D_{x,h}\right)D_{x,h}^n = 0.
\end{aligned}
\tag{3.21}
$$

□

Now, we are able to generalize the special differential operator $(d/dx)x(d/dx)$, stated as the Laguerre derivative in [11, 12], which appears in mathematical modelling of phenomena in viscous fluids and the oscillating chain in mechanics. Substituting the ordinary derivative and variable with the deformed one, we get the deformed Laguerre derivative

$$
\begin{aligned}
(D_{x,h}X_hD_{x,h})f(x) &= \left(\frac{d}{d_hx}\{x\}_h\frac{d}{d_hx}\right)f(x) \\
&= (1+hx)\frac{d}{dx}\left(\ln(1+hx)^{1/h}(1+hx)\frac{df(x)}{dx}\right).
\end{aligned}
\tag{3.22}
$$

Lemma 3.6. *For $x \in I$, $y \in \mathbb{R}$, and $k, n \in \mathbb{N}_0$, the following is valid:*

$$(D_{x,h}X_hD_{x,h})e_h(x,y) = y(1+yX_h)e_h(x,y), \tag{3.23}$$

$$(D_{x,h}X_hD_{x,h})^k\left(\frac{\{x\}_h^n}{n!}\right) = k!\binom{n}{k}\frac{\{x\}_h^{n-k}}{(n-k)!}. \tag{3.24}$$

Proof. With respect to Proposition 3.2, equality (3.10), and the product rule for $D_{x,h}$, we have

$$(D_{x,h}X_hD_{x,h})e_h(x,y) = D_{x,h}(\{x\}_hye_h(x,y)) = y(1+y\{x\}_h)e_h(x,y), \tag{3.25}$$

wherefrom we get the operational inscription. The second equality follows from the repeated application of (3.10). □

At last, we refer to the M and P operators as the descending (or lowering) and ascending (or raising) operators associated with the polynomial set $\{q_n\}_{n\in\mathbb{N}_0}$ if

$$M(q_n) = nq_{n-1}, \qquad P(q_n) = q_{n+1}. \tag{3.26}$$

Then, the polynomial set $\{q_n\}_{n\in\mathbb{N}_0}$ is called quasimonomial with respect to the operators M and P (see [16]).

It is easy to see that $D_{x,h}$ and X_h are the descending and ascending operators associated with the set of generalized monomial $\{x\}_h^n (n \in \mathbb{N}_0)$. Also, $D_{x,h}X_hD_{x,h}$ and $D_{x,h}^{-1}$ are the descending and ascending operators associated with the set of generalized monomial $(\{x\}_h^n/n!)$ $(n \in \mathbb{N}_0)$.

4. The Functional Sequence Induced by Iterated Deformed Laguerre Derivative

Let $h \neq 0$, $I = (-\infty, -1/h)$ for $h < 0$ or $I = (-1/h, +\infty)$ for $h > 0$ and $G = I \times \mathbb{R}^+$. We define functions $(x, y) \mapsto L_{n,h}(x, y)$ for $(x, y) \in G$ $(n \in \mathbb{N}_0)$ by the relation

$$L_{n,h}(x, y) = \frac{(-1)^n}{y^n} e_h(x, y)(D_{x,h}X_h D_{x,h})^n (e_h(x, -y)). \tag{4.1}$$

The first members of the functional sequence $\{L_{n,h}(x, y)\}_{n\in\mathbb{N}_0}$ are

$$\begin{aligned} L_{0,h}(x, y) &= 1, \\ L_{1,h}(x, y) &= 1 - y \ln\left((1 + hx)^{1/h}\right) = 1 - \{x\}_h y, \\ L_{2,h}(x, y) &= 2 - 4y\{x\}_h + y^2\{x\}_h^2. \end{aligned} \tag{4.2}$$

Lemma 4.1. *The function $L_{n,h}(x, y)$ $(n \in \mathbb{N}_0)$ is the polynomial of degree n in the deformed variable $y\{x\}_h = \ln((1 + hx)^{y/h}) = \ln e_h(x, y)$.*

Proof. Using equality (3.23), we have

$$(D_{x,h}X_h D_{x,h})e_h(x, -y) = y(y\{x\}_h - 1)e_h(x, -y). \tag{4.3}$$

With respect to (3.24), repeating the previous step, we get

$$(D_{x,h}X_h D_{x,h})^n (e_h(x, -y)) = y^n Q_n(y\{x\}_h)e_h(x, -y), \tag{4.4}$$

where Q_n is a monic polynomial of degree n. According to Proposition 2.1, we have

$$L_{n,h}(x, y) = (-1)^n Q_n(y\{x\}_h). \tag{4.5}$$

□

Theorem 4.2. *The functions $L_{n,h}(x, y)$ satisfy the next relation of orthogonality:*

$$J_{n,k} = \int_0^A L_{n,h}(x, y)\{x\}_h^k e_h(x, -y - h)dx = \frac{(-1)^n(n!)^2}{y^{n+1}}\delta_{kn}, \tag{4.6}$$

where $k, n \in \mathbb{N}_0$, $k \leq n$, and

$$A = \begin{cases} +\infty, & h > 0, \\ -\dfrac{1}{h}, & h < 0. \end{cases} \tag{4.7}$$

Proof. Substituting $L_{n,h}(x,y)$ in integral $J_{n,k}$, according to Proposition 2.1, we get

$$
\begin{aligned}
J_{n,k} &= \frac{(-1)^n}{y^n} \int_0^A e_h(x,y)(D_{x,h}X_hD_{x,h})^n(e_h(x,-y))\{x\}_h^k \, e_h(x,-y-h)dx \\
&= \frac{(-1)^n}{y^n} \int_0^A (D_{x,h}X_hD_{x,h})^n(e_h(x,-y))\{x\}_h^k \, e_h(x,-h)\,dx.
\end{aligned}
\tag{4.8}
$$

Since

$$
\begin{aligned}
(D_{x,h}X_hD_{x,h})^n &= (D_{x,h}X_hD_{x,h})(D_{x,h}X_hD_{x,h})^{n-1} \\
&= (1+hx)\frac{d}{dx}\{x\}_h\frac{d}{d_hx}\left(\frac{d}{d_hx}\{x\}_h\frac{d}{d_hx}\right)^{n-1}
\end{aligned}
\tag{4.9}
$$

and $e_h(x,-h) = (1+hx)^{-1}$, the integral becomes

$$
J_{n,k} = \frac{(-1)^n}{y^n}\int_0^A \frac{d}{dx}\left(\{x\}_h\frac{d}{d_hx}\left(\frac{d}{d_hx}\ \{x\}_h\ \frac{d}{d_hx}\right)^{n-1}(e_h(x,-y))\right)\{x\}_h^k\,dx.
\tag{4.10}
$$

Applying integration by parts twice and using relation (3.10), we get

$$
\begin{aligned}
J_{n,k} &= \left[\{x\}_h^{k+1}q(\{x\}_h)e_h(x,-y)\right]_{x=0}^{x=A} \\
&\quad + \frac{(-1)^{n+2}}{y^n}k^2\int_0^A\left(\frac{d}{d_hx}\{x\}_h\frac{d}{d_hx}\right)^{n-1}(e_h(x,-y))\{x\}_h^{k-1}e_h(x,-h)dx,
\end{aligned}
\tag{4.11}
$$

where q is a polynomial. Because of

$$
\lim_{x\nearrow A}\{x\}_h^m e_h(x,-y) = \lim_{x\nearrow A}\frac{\left(\ln\left((1+hx)^{1/h}\right)\right)^m}{(1+hx)^{y/h}} = 0 \quad (m\in\mathbb{N}_0),
\tag{4.12}
$$

we have

$$
J_{n,k} = \frac{(-1)^n}{y^n}k^2\int_0^A (D_{x,h}X_hD_{x,h})^{n-1}(e_h(x,-y))\{x\}_h^{k-1}e_h(x,-h)dx.
\tag{4.13}
$$

Repeating the procedure k times, we get

$$
J_{n,k} = \frac{(-1)^n}{y^n}(k!)^2\int_0^A (D_{x,h}X_hD_{x,h})^{n-k}(e_h(x,-y))e_h(x,-h)dx.
\tag{4.14}
$$

If $n > k$, then the following holds:

$$\begin{aligned} J_{n,k} &= \frac{(-1)^n}{y^n}(k!)^2 \int_0^A \frac{d}{dx}\left(\{x\}_h \frac{d}{d_h x}\left(\frac{d}{d_h x}\{x\}_h \frac{d}{d_h x}\right)^{n-k-1}\right)(e_h(x,-y))\,dx \\ &= \frac{(-1)^n}{y^n}(k!)^2 \left[\{x\}_h \frac{d}{d_h x}\left(\frac{d}{d_h x}\{x\}_h \frac{d}{d_h x}\right)^{n-k-1}(e_h(x,-y))\right]_{x=0}^{x=A} = 0. \end{aligned} \tag{4.15}$$

If $n = k$, then

$$\begin{aligned} J_{n,n} &= \frac{(-1)^n}{y^n}(n!)^2 \int_0^A e_h(x,-y)e_h(x,-h)\,dx \\ &= \frac{(-1)^n}{y^n}(n!)^2 \int_0^A e_h(x,-y-h)\,dx = \frac{(-1)^n}{y^{n+1}}(n!)^2. \end{aligned} \tag{4.16}$$

□

Notice that the orthogonality relation can be also written in the form

$$\int_0^A L_{n,h}(x,y)\ln^k(1+hx)\frac{dx}{(1+hx)^{y/h+1}} = \frac{(-1)^n(n!)^2h^n}{y^{n+1}}\delta_{kn} \quad (k,n \in \mathbb{N}_0,\ k \le n). \tag{4.17}$$

This orthogonality relation and other properties that will be proven indicate that the functions $L_{n,h}(x,y)$ are in close connection with the Laguerre polynomials. That is why we will call them the *deformed Laguerre polynomials*.

5. Properties of the Deformed Laguerre Polynomials

Let us recall that the Laguerre polynomials defined by [17]

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) = n!\sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{x^k}{k!} \quad (n \in \mathbb{N}_0) \tag{5.1}$$

satisfy the orthogonality relation

$$\int_0^\infty L_n(x)L_m(x)e^{-x}\,dx = (n!)^2\delta_{mn} \quad (m,n \in \mathbb{N}_0), \tag{5.2}$$

the three-term recurrence relations

$$L_{n+1}(x) + (x-2n-1)L_n(x) + n^2L_{n-1}(x) = 0 \quad (n \in \mathbb{N}), \tag{5.3}$$

and the differential equations of second order

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0 \quad (n \in \mathbb{N}). \tag{5.4}$$

Theorem 5.1. *The functions $L_{n,h}(x,y)(n \in \mathbb{N}_0)$ can be represented by*

$$L_{n,h}(x,y) = L_n(y\{x\}_h). \tag{5.5}$$

Proof. Having in mind that $L_{n,h}(x,y) = (-1)^n Q_n(y\{x\}_h)$, where Q_n is a monic polynomial of degree n, and changing variable x by $t = y\{x\}_h$ in integrals $J_{n,k}$ for $k \le n$, we have

$$(n!)^2 \delta_{kn} = (-1)^n y^{n+1} J_{n,k} = \int_0^{+\infty} Q_n(t) t^k e^{-t}\, dt. \tag{5.6}$$

It is a well-known orthogonality relation for the Laguerre polynomials. That is why $Q_n(t) = c_n L_n(t)$, where c_n = const. Since Q_n is monic and

$$\int_0^{+\infty} L_n(t) t^n e^{-t}\, dt = (-1)^n (n!), \tag{5.7}$$

it must be that $c_n = (-1)^n$ and therefore $L_{n,h}(x,y) = L_n(t) = L_n(y\{x\}_h)$. □

The next corollaries express two concepts of orthogonality of these functions.

Corollary 5.2. *For $m, n \in \mathbb{N}_0$, the following is valid:*

$$\int_0^A L_{m,h}(x,y) L_{n,h}(x,y) e_h(x, -y-h)\, dx = \frac{(n!)^2}{y} \delta_{mn} \quad (y > 0), \tag{5.8}$$

where

$$A = \begin{cases} +\infty, & h > 0, \\ -\dfrac{1}{h}, & h < 0, \end{cases}$$
$$\int_0^{+\infty} L_{m,h}(x,y) L_{n,h}(x,y) e_h(x,-y)\, dy = \frac{(n!)^2}{\{x\}_h} \delta_{mn} \quad (x \in I \setminus \{0\}). \tag{5.9}$$

From this close connection of functions $L_{n,h}(x,y)$ with the Laguerre polynomials, their properties, as the summation formula, recurrence relation, or differential equation, follow immediately.

Corollary 5.3. *The functions $L_{n,h}(x,y)$ $(n \in \mathbb{N}_0)$ have the next hypergeometric representation:*

$$L_{n,h}(x,y) = n!\ {}_1F_1\left(\begin{array}{c} -n \\ 1 \end{array} \middle| \, y\{x\}_h \right) = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{y^k \{x\}_h^k}{k!}. \tag{5.10}$$

Corollary 5.4. *The function $x \mapsto L_{n,h}(x,y)(y>0)$ is a solution of the differential equation*

$$(1+hx)^2\ln(1+hx)\frac{d^2}{dx^2}(Z(x)) + (1+hx)\big(h-(y-h)\ln(1+hx)\big)\frac{d}{dx}(Z(x)) + yhnZ(x) = 0, \tag{5.11}$$

or, in the other form,

$$\frac{d^2}{d_hx^2}(Z(x)) + \big(1-y\{x\}_h\big)\frac{d}{d_hx}(Z(x)) + ynZ(x) = 0. \tag{5.12}$$

Proof. The first form of equation is obtained from the differential equation of the Laguerre polynomials and Theorem 5.1. For the second one, it is enough to notice that

$$\frac{d^2}{d_hx^2}(Z(x)) = \left((1+hx)\frac{d}{dx}\right)^2(Z(x)) = (1+hx)^2Z''(x) + h(1+hx)Z'(x). \tag{5.13}$$

□

Corollary 5.5. *The function $y \mapsto L_{n,h}(x,y)(x\in I)$ is a solution of the differential equation*

$$\frac{d^2}{dy^2}(Z(y)) + (1-y\{x\}_h)\{x\}_h\frac{d}{dy}(Z(y)) + n\{x\}_h^2Z(y) = 0. \tag{5.14}$$

Theorem 5.6. *The sequence $\{L_{n,h}(x,y)\}_{n\in\mathbb{N}_0}$ has the following generating function:*

$$\frac{1}{1-t}e_h\left(x,-\frac{yt}{1-t}\right) = \sum_{n=0}^{\infty}L_{n,h}(x,y)\frac{t^n}{n!} \quad (x\in I,\ |t|<1). \tag{5.15}$$

Proof. Let

$$\mathcal{G}_h(x,y,t) = \sum_{n=0}^{\infty}L_{n,h}(x,y)\frac{t^n}{n!}. \tag{5.16}$$

Notice that

$$t\frac{\partial}{\partial t}(\mathcal{G}_h(x,y,t)) = \sum_{n=0}^{\infty}nL_{n,h}(x,y)\frac{t^n}{n!}. \tag{5.17}$$

According to Theorem 5.1 and recurrence relation (5.3), by summation we get

$$\begin{gathered}\sum_{n=0}^{\infty}\Big(L_{n+1,h}(x,y) + (y\{x\}_h-2n-1)L_{n,h}(x,y) + n^2L_{n-1,h}(x,y)\Big)\frac{t^n}{n!} = 0,\\ \sum_{n=0}^{\infty}L_{n+1,h}(x,y)\frac{t^n}{n!} + (y\{x\}_h-1)\sum_{n=0}^{\infty}L_{n,h}(x,y)\frac{t^n}{n!}\\ -2\sum_{n=0}^{\infty}nL_{n,h}(x,y)\frac{t^n}{n!} + t\sum_{n=0}^{\infty}(n+1)L_{n,h}(x,y)\frac{t^n}{n!} = 0.\end{gathered} \tag{5.18}$$

According to (5.17), we have

$$(1-t)^2\frac{\partial}{\partial t}(\mathcal{G}_h(x,y,t)) + (y\{x\}_h - 1 + t)\mathcal{G}_h(x,y,t) = 0. \tag{5.19}$$

Solving the obtained differential equation with respect to the initial condition $\mathcal{G}_h(x,y,0) = L_{0,h}(x,y) = 1$, we get

$$\mathcal{G}_h(x,y,t) = \frac{1}{1-t}\, e^{-yt\{x\}_h/(1-t)} = \frac{1}{1-t}\, e_h\left(x, -\frac{yt}{1-t}\right). \tag{5.20}$$

□

Theorem 5.7. *The functions $\{L_{n,h}(x,y)\}$ have the following differential properties:*

$$nyL_{n-1,h}(x,y) = n\frac{d}{d_hx}(L_{n-1,h}(x,y)) - \frac{d}{d_hx}(L_{n,h}(x,y)), \tag{5.21}$$

$$\frac{d}{d_hx}(L_{n+1,h}(x,y)) = -(n+1)!y\sum_{k=0}^{n}\frac{L_{k,h}(x,y)}{k!}, \tag{5.22}$$

$$n\{x\}_hL_{n,h}(x,y) = n\frac{\partial}{\partial y}(L_{n-1,h}(x,y)) - \frac{\partial}{\partial y}(L_{n,h}(x,y)), \tag{5.23}$$

$$\frac{\partial}{\partial y}(L_{n+1,h}(x,y)) = -(n+1)!\{x\}_h\sum_{k=0}^{n}\frac{L_{k,h}(x,y)}{k!}. \tag{5.24}$$

Proof. Applying operator $D_{x,h} = d/d_hx$ to equality (5.15), according to Proposition 3.2, we get

$$\frac{-yt}{(1-t)^2}\, e_h\left(x, -\frac{yt}{1-t}\right) = \sum_{n=0}^{\infty}\frac{d}{d_hx}(L_{n,h}(x,y))\frac{t^n}{n!}, \tag{5.25}$$

that is,

$$\frac{-yt}{1-t}\sum_{n=0}^{\infty}L_{n,h}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{d}{d_hx}(L_{n,h}(x,y))\frac{t^n}{n!}. \tag{5.26}$$

Multiplying by $1-t$ and comparing coefficients, we get equality (5.21). In a similar way, using operator $\partial/\partial y$, we get equality (5.23). Equalities (5.22) and (5.24) can be obtained comparing coefficients of powers of t, but using expansion

$$\frac{t}{1-t} = \sum_{k=0}^{\infty}t^{k+1}. \tag{5.27}$$

Theorem 5.8. *For functions $\{L_{n,h}(x,y)\}$, the next addition formulas are valid:*

$$L_{n,h}(x_1 \oplus_h x_2, y) = \sum_{k=0}^{n} \binom{n}{k} L_{k,h}(x_1, y) L_{n-k,h}(x_2, y),$$
$$L_{n,h}(x, y_1 + y_2) = \sum_{k=0}^{n} \binom{n}{k} L_{n-k,h}(x, y_1)\left(L_{k,h}(x; y_2) - kL_{k-1,h}(x; y_2)\right). \tag{5.28}$$

Proof. We get the first addition formula from Theorem 5.1, equalities (2.11)–(2.14), and the addition formula for the Laguerre polynomials [18]:

$$L_n(x_1 + x_2) = \sum_{k=0}^{n} \binom{n}{k} L_k(x_1) L_{n-k}(x_2). \tag{5.29}$$

For the second formula, we consider generating function. According to (5.15) and Proposition 2.1, we have

$$\mathcal{G}_h(x, y_1 + y_2, t) = (1-t)\mathcal{G}_h(x, y_1, t)\mathcal{G}_h(x, y_2, t), \tag{5.30}$$

that is,

$$\sum_{n=0}^{\infty} \frac{L_{n,h}(x, y_1 + y_2)}{n!} t^n = (1-t) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{L_{n-k,h}(x, y_1) L_{k,h}(x, y_2)}{(n-k)!k!} t^n. \tag{5.31}$$

Comparing coefficients of t^n, we get the required equality. □

6. The Deformed Laguerre Polynomials in the Context of Operational Calculus

In this section, we consider the deformed Laguerre polynomials from the operational aspect (see [16, 19]). Let us denote

$$\widehat{L}_{n,h}(x,y) = \frac{1}{n!} L_{n,h}(x,y). \tag{6.1}$$

Theorem 6.1. *The polynomials $\widehat{L}_{n,h}(x,y)$ have the following operational representations:*

$$\widehat{L}_{n,h}(x,y) = \left(1 - yD_{x,h}^{-1}\right)^n (1), \qquad \widehat{L}_{n,h}(x,y) = (D_{x,h} - y)^n \left(\frac{\{x\}_h^n}{n!}\right). \tag{6.2}$$

Proof. According to Corollary 5.3 and Lemma 3.3, we have

$$\begin{aligned}
\widehat{L}_{n,h}(x,y) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{y^k\{x\}_h^k}{k!} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}y^kD_{x,h}^{-k}(1) \\
&= (-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(yD_{x,h}^{-1}\right)^k(1) = (-1)^n\left(yD_{x,h}^{-1}-1\right)^n(1) \\
&= \left(1-yD_{x,h}^{-1}\right)^n(1).
\end{aligned}\tag{6.3}$$

In a similar way,

$$\begin{aligned}
(D_{x,h}-y)^n\left(\frac{\{x\}_h^n}{n!}\right) &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}y^{n-k}D_{x,h}^k\left(\frac{\{x\}_h^n}{n!}\right) \\
&= \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!(n-k)!}y^{n-k}k!\binom{n}{k}\{x\}_h^{n-k} \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{n-k}\frac{y^{n-k}\{x\}_h^{n-k}}{(n-k)!} = \widehat{L}_{n,h}(x,y).
\end{aligned}\tag{6.4}$$

□

Theorem 6.2. *The polynomial set $\{\widehat{L}_{n,h}(x,y)\}_{n\in\mathbb{N}_0}$ is quasimonomial associated to the descending and ascending operators M_x and P_x, respectively:*

$$M_x = 1-yD_{x,h}^{-1}, \qquad P_x = -\frac{1}{y}D_{x,h}X_hD_{x,h}.\tag{6.5}$$

Proof. Using the previous theorem, we show that M_x is the descending operator for $\{\widehat{L}_{n,h}(x,y)\}$:

$$M_x\left(\widehat{L}_{n,h}(x,y)\right) = \left(1-yD_{x,h}^{-1}\right)\left(1-yD_{x,h}^{-1}\right)^n(1) = \left(1-yD_{x,h}^{-1}\right)^{n+1}(1) = \widehat{L}_{n+1,h}(x,y).\tag{6.6}$$

Also, P_x is the ascending operator because of

$$\begin{aligned}
P_x\left(\widehat{L}_{n,h}(x,y)\right) &= \left(-\frac{1}{y}D_{x,h}X_hD_{x,h}\right)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{y^k\{x\}_h^k}{k!}\right) \\
&= -\frac{1}{y}D_{x,h}\left(\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{y^k\{x\}_h^k}{(k-1)!}\right) = \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{k!(n-k)!}\frac{ky^{k-1}\{x\}_h^{k-1}}{(k-1)!} \\
&= n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{y^k\{x\}_h^k}{k!} = n\widehat{L}_{n-1,h}(x,y).
\end{aligned}\tag{6.7}$$

Theorem 6.3. *For $\widehat{L}_{n,h}(x,y)$, the following is valid:*

$$\widehat{L}_{n,h}(x,y) = \exp\left(-\frac{1}{y}D_{x,h}X_hD_{x,h}\right)\left(\frac{(-y\{x\}_h)^n}{n!}\right). \tag{6.8}$$

Proof. Using the formal expansion of the exponential function and Lemma 3.6, we have

$$\begin{aligned}
&\exp\left(-\frac{1}{y}D_{x,h}X_hD_{x,h}\right)\left(\frac{(-y\{x\}_h)^n}{n!}\right)\\
&\quad=\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{y}D_{x,h}X_hD_{x,h}\right)^k\left(\frac{(-1)^n y^n\{x\}_h^n}{n!}\right)\\
&\quad=\sum_{k=0}^{\infty}(-1)^{n-k}\frac{y^{n-k}}{k!}(D_{x,h}X_hD_{x,h})^k\left(\frac{\{x\}_h^n}{n!}\right)\\
&\quad=\sum_{k=0}^{n}(-1)^{n-k}\frac{y^{n-k}}{k!}k!\binom{n}{k}\frac{\{x\}_h^{n-k}}{(n-k)!}=\sum_{k=0}^{n}(-1)^k y^k\binom{n}{k}\frac{\{x\}_h^k}{k!},
\end{aligned} \tag{6.9}$$

which, according to Corollary 5.3, proves the statement. □

Remark 6.4. When $h \to 0$ and $y = 1$, all properties of functions $L_{n,h}(x,y)$ give corresponding ones for the Laguerre polynomials (see, e.g., [12, 19, 20]).

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