The deformed exponential functions of two variables in the context of various statistical mechanics

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Abstract. In the recent development in various disciplines of physics, it is noted the need for including the deformed versions of the exponential functions. In last two decades, the Tsallis and Kaniadakis versions have found a lot of applications. In this paper, we consider the deformations which have two purposes. First, we introduce them like beginning of a more general mathematical approach where the Tsallis and Kaniadakis exponential functions are the special cases. Then, we wish to pay attention to the mathematical community that they have a lot of interesting properties from mathematical point of view and possibilities in applications. Really, we will show the differential and difference properties of our deformations which are important for the formation and explanation of continuous and discrete models of numerous phenomena.

Keywords: Exponential function, differential operator, difference operator.

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1 Introduction

In last quarter of seventeenth century, solving of the concrete real problems leaded to the definition of exponential function. New circumstances and challenges in the twentieth century required its generalizations and deformations. One–parameter deformations of exponential function have been proposed in the context of non-extensive statistic mechanics (see [1, 2, 3, 4]) relativistic statistical theory (see, for example, [5, 6]) and quantum–group theory [7].

C. Tsallis introduced an analogue of exponential function [1] in 1988. Afterwards, considering a connection between the generalized entropy and theory of quantum groups, S. Abe defined in 1997 another deformation in [7], and, more recently, G. Kaniadakis [5] proposed in 2001 a new one–parameter deformation for the exponential function. These deformations attracted attention of researchers from various scientific fields because of their successful role in description of fractal structured systems, non-regular diffusion, thermodynamical and gravitational–like systems, optimization algorithm, statistical conclusions and probability theory, several complex systems, etc. (see [2, 8, 9]).

Deformations of the exponential functions are considered in three main (complementary) directions: Formal mathematical developments [1, 2, 5, 6, 10]; observation of consistent concordance with experimental (or natural) behavior [5]; and theoretical physical developments [7]. A very interesting discussion on justification for introducing the generalizations of known functions can be found in [11].

In this paper, using a formal mathematical approach, we introduce two variants of the deformed exponential function of two variables. The deformations proposed have differential and difference properties that permit to express discrete and continual behavior by the same (see [12]). In these functions, well–known generalizations and deformations can be viewed as the special cases.

The paper is organized as follows. After sections devoted to introduction and preliminaries, we introduce the deformed exponential functions of two variables in the third and fourth sections. In the following two sections, we examine their difference and differential properties. We prove that these functions appear as the eigenfunctions of the difference and differential operators. Finally, in the last section, we focus on their application in growth models in the population dynamics and economy.

2 Preliminaries: powers and differences

Let $h \in \mathbb{R} \setminus \{0\}$. The generalized integer powers of real numbers have an important role in modern theoretical considerations. In that manner, we first introduce *backward and forward integer power* given by

$$\begin{aligned} z^{(0,h)} &= 1, \qquad z^{(n,h)} = \prod_{k=0}^{n-1}(z-kh) \quad (n \in \mathbb{N}), \\ z^{[0,h]} &= 1, \qquad z^{[n,h]} = \prod_{k=0}^{n-1}(z+kh) \quad (n \in \mathbb{N}). \end{aligned}$$

The *central integer power* is defined as

$$z^{\langle 0,h\rangle} = 1,$$
$$z^{\langle n,h\rangle} = \begin{cases} \prod_{k=0}^{m-1} (z-2kh)(z+2kh) & (n=2m,\ m\in\mathbb{N}), \\ z\prod_{k=0}^{m-1} \big(z-(2k+1)h\big)\big(z+(2k+1)h\big) & (n=2m+1,\ m\in\mathbb{N}_0). \end{cases}$$

For $n \in \mathbb{N}$, a relationship with the previously defined generalized powers is given by

$$z^{(n,h)} = z^{[n,-h]}, \qquad z^{\langle n,h\rangle} = z^{\langle n,-h\rangle}, \tag{1}$$

and

$$z^{\langle n,h\rangle} = \begin{cases} z^{(m,2h)}\ z^{[m,2h]} & (n=2m,\ m\in\mathbb{N}), \\ z\ (z-h)^{(m,2h)}(z+h)^{[m,2h]} & (n=2m+1,\ m\in\mathbb{N}_0), \end{cases}$$

$$z^{\langle n,h\rangle} = \ z\big(z+(n-2)h\big)^{(n-1,\ 2h)},$$
$$z^{\langle 2m,h\rangle} z^{\langle 2m+1,h\rangle} = z\ z^{(2m,h)}\ z^{[2m,h]}.$$

Consider the h–difference operators

$$\begin{aligned} \Delta_{z,h} f(z) &= \frac{f(z+h)-f(z)}{h}\ , \\ \nabla_{z,h} f(z) &= \frac{f(z)-f(z-h)}{h}\ , \\ \delta_{z,h} f(z) &= \frac{f(z+h)-f(z-h)}{2h}\ . \end{aligned}$$

Notice that

$$\nabla_{z,h} f(z) = \Delta_{z,-h} f(z) = \Delta_{z,h} f(z-h), \quad \delta_{z,-h} f(z) = \delta_{z,h} f(z).$$

We can prove that their acting on integer generalized powers is given by:

$$\Delta_{z,h}\, z^{(n,h)} = nz^{(n-1,h)}, \quad \nabla_{z,h}\, z^{[n,h]} = nz^{[n-1,h]}, \quad \delta_{z,h}\, z^{\langle n,h\rangle} = nz^{\langle n-1,h\rangle}. \tag{2}$$

3 The deformed exponential functions of the Tsallis type

Let $h \in \mathbb{R} \setminus \{0\}$. We define a function $(x,y) \mapsto e_h(x,y)$ by

$$e_h(x,y) = (1+hx)^{y/h} \qquad (x \in \mathbb{C} \setminus \{-1/h\},\ y \in \mathbb{R}). \tag{3}$$

Since

$$\lim_{h\to 0} e_h(x,y) = e^{xy}\ ,$$

this function can be viewed as an one–parameter deformation of the exponential function of two variables.

If $h = 1 - q$ ($q \neq 1$) and $y = 1$, the function (3) becomes

$$e_{1-q}(x, 1) = \big(1 + (1-q)x\big)^{1/(1-q)},$$

i.e., $e_{1-q}(x, 1) = e_q^x$, where e_q^x is Tsallis q–exponential function [1] defined by

$$e_q^x = \begin{cases} \big(1 + (1-q)x\big)^{1/(1-q)}, & 1 + (1-q)x > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (x \in \mathbb{R}).$$

If $h = p - 1$ ($p \neq 1$) and $x = 1$, the function (3) becomes

$$e_{p-1}(1, y) = p^{y/(p-1)},$$

i.e. function considered for a generalization of the standard exponential function in the context of quantum group formalism [13].

Remark 3.1 The function $e_h(x, y)$ can be viewed as the scaled Tsallis exponential function:

$$e_h(x, y) = e_{q'}^{x'} \quad \text{with} \quad x' = xy \quad \text{with} \quad q' = 1 - h/y\ .$$

Example 3.1 The function $e_h(x, y)$ determines a surface shown in Figure 1. Bold-emphasized line is e^x in the first, and e^y in the second part.

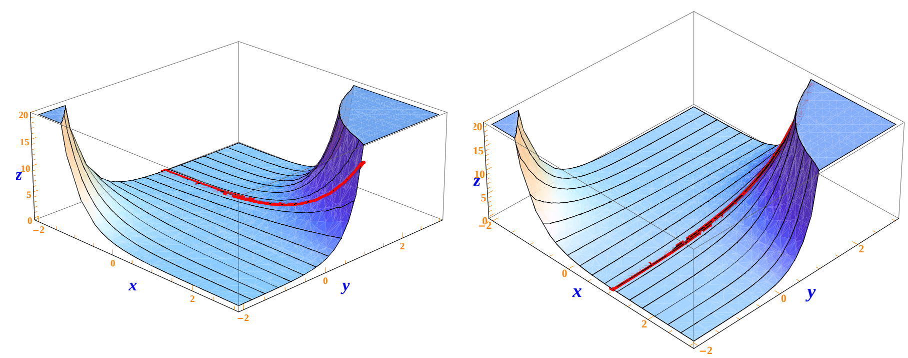

Figure 1: The behavior of $e_h(x, y)$ for $h = 0.05$. The level lines show cases:
a) $y =$const; b) $x =$const.

Notice that function (3) can be written in the form

$$e_h(x, y) = \exp\Big(\frac{y}{h}\ \ln(1 + hx)\Big).$$

Hence, we can define a deformation function $x \mapsto \{x\}_h$ by

$$\{x\}_h = \frac{1}{h}\ \ln(1+hx) \qquad (x \in \mathbb{C} \setminus \{-1/h\}). \tag{4}$$

Thus, the following holds:

$$e_h(x,y) = e^{\{x\}_h\, y}. \tag{5}$$

We can show that the function (3) holds on some basic properties of exponential function.

Proposition 3.1 *For $x \in \mathbb{C} \setminus \{-1/h\}$ and $y \in \mathbb{R}$ the following holds:*

$$\begin{aligned} e_h(x,y) &> 0 \qquad (x < -1/h \quad \text{for } h<0 \quad \text{or} \quad x > -1/h \quad \text{for } h>0),\\ e_h(0,y) &= e_h(x,0) = 1,\\ e_{-h}(x,y) &= e_h(-x,-y) \qquad (x \neq 1/h),\\ e_h(x,y_1+y_2) &= e_h(x,y_1)e_h(x,y_2). \end{aligned}$$

Notice that an additional property is true with respect to the second variable only. However, with respect to the first variable, the following holds:

$$e_h(x_1,y)e_h(x_2,y) = e_h\big(x_1+x_2+hx_1x_2, y\big)\ .$$

This equality suggests us to introduce a generalization of the sum operation [1] (see [3], [4])

$$x_1 \oplus_h x_2 = x_1 + x_2 + hx_1x_2\ . \tag{6}$$

This operation is commutative, associative and 0 is its identity. For $x \neq -1/h$, the $\ominus_h$– inverse exists as

$$\ominus_h x = \frac{-x}{1+hx}$$

and $x \oplus_h (\ominus_h x) = 0$ is valid. Hence, $(I, \oplus_h)$ is an Abelian group, where $I = (-\infty, -1/h)$ for $h<0$ or $I = (-1/h, +\infty)$ for $h>0$ (see [14]). In this way, the $\ominus_h$–subtraction can be defined by

$$x_1 \ominus_h x_2 = x_1 \oplus_h (\ominus_h x_2) = \frac{x_1 - x_2}{1+hx_2} \qquad \Big(x_2 \neq -\frac{1}{h}\Big). \tag{7}$$

Due to (4), we can prove the next equality:

$$\{x_1\}_h + \{x_2\}_h = \{x_1 \oplus_h x_2\}_h \qquad (x_1, x_2 \in I). \tag{8}$$

Now, we will prove several new properties of the function (3).

Theorem 3.1 *For $x_1, x_2 \in \mathbb{C} \setminus \{-1/h\}$ and $y \in \mathbb{R}$, the following is valid:*

$$\begin{aligned} e_h(x_1 \oplus_h x_2, y) &= e_h(x_1,y)e_h(x_2,y),\\ e_h(x_1 \ominus_h x_2, y) &= e_h(x_1,y)e_h(x_2,-y). \end{aligned}$$

[1]Notice that generalized summation was already used in [15, 16, 17] for the analyzing of functions.

Proof. The first equality follows immediately from (3) and (6). For the second one, we can notice the following:

$$e_h(\ominus_h x, y) = e_h\Big(\frac{-x}{1+hx}, y\Big) = \Big(1 - \frac{hx}{1+hx}\Big)^{y/h} = \frac{1}{(1+hx)^{y/h}} = e_h(x, -y). \ \square$$

The convenience of new notation can be seen in the next example. Using it, we will express in an elegant way the deformation of the special number sequence.

Example 3.2 Let us consider the generating function

$$\frac{e_h(\Phi, t) - e_h(-\Phi^{-1}, t)}{\{\Phi\}_h - \{-\Phi^{-1}\}_h} = \sum_{n=0}^{\infty} F_{h,n} \frac{t^n}{n!} \qquad \left(\Phi = \frac{1+\sqrt{5}}{2}\right).$$

We call $\{F_{h,n}\}$ the *h-Fibonacci numbers*. They have the following explicit form and limits:

$$F_{h,n} = \frac{\{\Phi\}_h^n - \{-\Phi^{-1}\}_h^n}{\{\Phi\}_h - \{-\Phi^{-1}\}_h}, \quad \lim_{n\to\infty} F_{h,n} = F_n, \quad \lim_{n\to\infty} \frac{F_{h,n}}{F_{h,n-1}} = \{\Phi\}_h.$$

The next recurrence relation is valid

$$F_{h,n+2} = \{1-h\}_h F_{h,n+1} - \big(\{\Phi\}_h \cdot \{-\Phi^{-1}\}_h\big)\, F_{h,n} \quad (n \in \mathbb{N}).$$

For above, the property (8) is used: $\{\Phi\}_h + \{-\Phi^{-1}\}_h = \{1-h\}_h$.

4 The deformed exponential functions of Kaniadakis type

Let us define a function $(x, y) \mapsto \exp_h(x, y)$ by

$$\exp_h(x, y) = \big(hx + \sqrt{1 + h^2x^2}\big)^{y/h} \qquad (x \in \mathbb{C},\ y \in \mathbb{R}). \tag{9}$$

Since

$$\lim_{h\to 0} \exp_h(x, y) = e^{xy},$$

this function can be viewed as an one–parameter deformation of the exponential function with two variables.

If $h = \kappa$ and $y = 1$, the function (9) becomes κ–exponential function

$$\exp_\kappa(x, 1) = \exp_{\{\kappa\}}(x) = \big(\sqrt{1+\kappa^2x^2} + \kappa x\big)^{1/\kappa},$$

introduced by Kaniadakis in [5, 6].

Remark 4.1 The function $\exp_h(x, y)$ can be viewed as a scaled Kaniadakis exponential:

$$\exp_h(x, y) = \exp_{\kappa'}(x') \quad \text{with} \quad x' = xy \quad \text{and} \quad \kappa' = h/y.$$

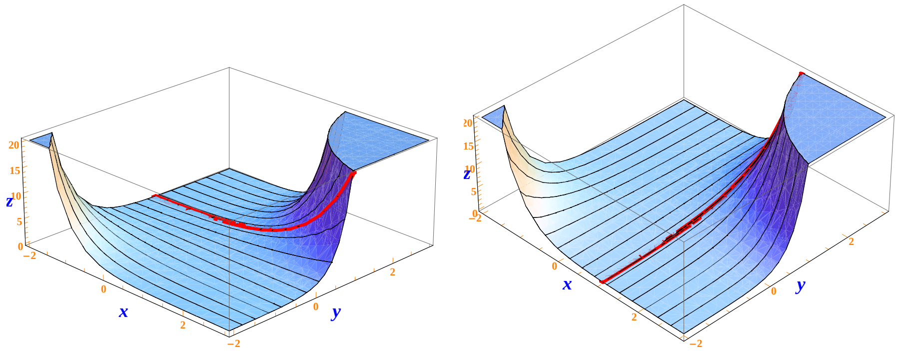

Figure 2: The behavior of $\exp_h(x,y)$ for $h=0.05$. The level lines show cases: a) $y=$const; b) $x=$const.

Example 4.1 The surface determined by $\exp_h(x,y)$ for very small h is shown on Figure 2. The bold–emphasized curve is e^x in the first, and e^y in the second part of the figure.

The relationhip between functions (3) and (9) is given by

$$\exp_h(x,y) = e_h\Big(x - \frac{1-\sqrt{1+h^2x^2}}{h}\ ,\ y\Big).$$

Since

$$\operatorname{arcsinh}(hx) = \ln(hx + \sqrt{1+h^2x^2}), \tag{10}$$

(9) can be written in the form

$$\exp_h(x,y) = \exp\Big(\frac{y}{h}\,\operatorname{arcsinh} hx\Big).$$

In [6], the deformation function $x \mapsto \{x\}^h$ was defined by

$$\{x\}^h = \frac{1}{h}\,\operatorname{arcsinh} hx \qquad (x \in \mathbb{C}). \tag{11}$$

Now, function (9) can be written as

$$\exp_h(x,y) = e^{\{x\}^h y}. \tag{12}$$

We adduce the main properties of the introduced deformed exponential function without the proof.

Proposition 4.1 *For $x \in \mathbb{C}$ and $y \in \mathbb{R}$ the following holds:*

$$\begin{aligned} \exp_h(x,y) &> 0 \qquad (x \in \mathbb{R}),\\ \exp_h(0,y) &= \exp_h(x,0) = 1,\\ \exp_{-h}(x,y) &= \exp_h(x,y),\\ \exp_h(x,y_1+y_2) &= \exp_h(x,y_1)\exp_h(x,y_2). \end{aligned}$$

This function holds on additional property with respect to the second variable only. However, according to (10) we have:

$$\exp_h(x_1,y)\exp_h(x_2,y)=\exp_h\Big(x_1\sqrt{1+h^2x_2^2}+x_2\sqrt{1+h^2x_1^2},y\Big).$$

This suggests that we introduce another generalization of sum operation (see [5]):

$$x_1\oplus^h x_2=x_1\sqrt{1+h^2x_2^2}+x_2\sqrt{1+h^2x_1^2}\ . \tag{13}$$

The operation $\oplus^h-$ sum is commutative, associative, its identity is 0 and $\oplus^h-$ inverse for $x\in\mathbb{R}$ is $-x$. Thus, $(\mathbb{R},\oplus^h)$ is an Abelian group, and $\ominus^h-$ subtraction can be defined by

$$x_1\ominus^h x_2=x_1\oplus^h(-x_2)=x_1\sqrt{1+h^2x_2^2}-x_2\sqrt{1+h^2x_1^2}\ . \tag{14}$$

Related to (11), we can prove the next equality:

$$\{x_1\}^h+\{x_2\}^h=\{x_1\oplus^h x_2\}^h.$$

With respect to the operation $\oplus^h$, the function (9) has the following properties:

Theorem 4.1 *For $x_1,x_2\in\mathbb{C}$ and $y\in\mathbb{R}$, the following is valid:*

$$\exp_h(x_1\oplus^h x_2,y)=\exp_h(x_1,y)\exp_h(x_2,y),$$
$$\exp_h(x_1\ominus^h x_2,y)=\exp_h(x_1,y)\exp_h(-x_2,y)=\exp_h(x_1,y)\exp_h(x_2,-y).$$

Example 4.2 Starting from the generating function

$$\frac{\exp_h(\Phi,t)-\exp_h(-\Phi^{-1},t)}{\{\Phi\}^h-\{-\Phi^{-1}\}^h}=\sum_{n=0}^{\infty}\mathrm{Fib}_{h,n}\frac{t^n}{n!}\qquad\left(\Phi=\frac{1+\sqrt{5}}{2}\right),$$

we can get the other class of the *h-Fibonacci numbers*, $\{\mathrm{Fib}_{h,n}\}$. For this sequence the following is valid:

$$\mathrm{Fib}_{h,n}=\frac{\big(\{\Phi\}^h\big)^n-\big(\{-\Phi^{-1}\}^h\big)^n}{\{\Phi\}^h+\{\Phi^{-1}\}^h},\ \lim_{n\to\infty}\mathrm{Fib}_{h,n}=F_n,\ \lim_{n\to\infty}\frac{\mathrm{Fib}_{h,n}}{\mathrm{Fib}_{h,n-1}}=\{\Phi\}^h,$$

$$\mathrm{Fib}_{h,n+2}=\big(\{\Phi\}^h-\{\Phi^{-1}\}^h\big)\mathrm{Fib}_{h,n+1}+\big(\{\Phi\}^h\cdot\{\Phi^{-1}\}^h\big)\mathrm{Fib}_{h,n}\quad(n\in\mathbb{N}).$$

5 Expansions and difference properties of deformed exponential functions

In this section we consider the expansions of the introduced deformed exponential functions. Related to these expansions, we show that functions $e_h(x,y)$ and $e_{-h}(x,y)$ are eigenfunctions of the operators $\Delta_{y,h}$ and $\nabla_{y,h}$ with the eigenvalue x. Likewise the function $\exp_h(x,y)$ is an eigenfunction of operator $\delta_{y,h}$ with eigenvalue x.

Theorem 5.1 *For functions $(x,y)\mapsto e_h(x,y)$ and $(x,y)\mapsto e_{-h}(x,y)$, the following representations hold respectively:*

$$\begin{aligned} e_h(x,y) &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n y^{(n,h)} \qquad (|hx|<1), &(15)\\ e_{-h}(x,y) &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n y^{[n,h]} \qquad (|hx|<1). &(16)\end{aligned}$$

Proof. With respect to well–known expansion

$$(1+z)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}z^n \quad (|z|<1,\ \alpha\in\mathbb{R}),$$

and the relation

$$\binom{z/h}{n}=\frac{z(z-h)\cdots\big(z-(n-1)h\big)}{h^n\ n!}=\frac{z^{(n,h)}}{h^n\ n!}\ ,$$

the following holds:

$$(1+hx)^{y/h}=\sum_{n=0}^{\infty}\binom{y/h}{n}h^n x^n=\sum_{n=0}^{\infty}\frac{y^{(n,h)}}{h^n\ n!}h^n x^n \qquad (|hx|<1).$$

Hence, we get the required expansion (15) for $e_h(x,y)$. Using (1), we obtain the expansion (16) for $e_{-h}(x,y)$. $\square$

Theorem 5.2 *The functions $y\mapsto e_h(x,y)$ and $y\mapsto e_{-h}(x,y)$ are the eigenfunctions of operators $\Delta_{y,h}$ and $\nabla_{y,h}$ respectively, with the eigenvalue x.*

Proof. The statement follows from (2) and the expansion (15). Hence, the function $f(y)=e_h(x,y)$ satisfies the difference equation

$$\Delta_{y,h}\ f(y)=x\ f(y).$$

In a similar way, using (2) and (16), we can show that $f(y)=e_{-h}(x,y)$ satisfies the difference equation

$$\nabla_{y,h}\ f(y)=x\ f(y)\ . \quad \square$$

Theorem 5.3 *The function $y\mapsto \exp_h(x,y)$ is an eigenfunction of operator $\delta_{y,h}$ with eigenvalue x.*

Proof. For function $\exp_h(x,y)$ the following is valid:

$$\begin{aligned}\delta_{y,h}\exp_h(x,y) &= \frac{1}{2h}\Big(\big(hx+\sqrt{1+h^2x^2}\big)^{(y+h)/h}-\big(hx+\sqrt{1+h^2x^2}\big)^{(y-h)/h}\Big)\\ &= \frac{\big(hx+\sqrt{1+h^2x^2}\big)^{\frac{y}{h}-1}}{2h}\Big(\big(hx+\sqrt{1+h^2x^2}\big)^2-1\Big)\\ &= x\exp_h(x,y)\ .\end{aligned}$$

Therefore, function $f(y) = \exp_h(x,y)$ satisfies difference equation

$$\delta_{y,h} f(y) = x f(y). \quad \Box$$

Theorem 5.4 *The function* $(x,y) \mapsto \exp_h(x,y)$ *can be represented as*

$$\exp_h(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n y^{\langle n,h \rangle} \tag{17}$$

Proof. Consider the expansion of function (9) in the form

$$\exp_h(x,y) = \sum_{n=0}^{\infty} \frac{c_n(y,h)}{n!} x^n.$$

Inasmuch as

$$\delta_{y,h} \exp_h(x,y) = x \exp_h(x,y),$$

it follows

$$\begin{aligned} \delta_{y,h} \sum_{n=0}^{\infty} \frac{c_n(y,h)}{n!} x^n &= \sum_{n=0}^{\infty} \frac{\delta_{y,h} c_n(y,h)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{c_n(y,h)}{n!} x^{n+1} = \sum_{n=1}^{\infty} \frac{c_{n-1}(y,h)}{(n-1)!} x^n. \end{aligned}$$

Therefore the coefficients have to be

$$\delta_{y,h} c_0(y) = 0, \qquad \delta_{y,h} c_n(y,h) = n c_{n-1}(y,h),$$

wherefrom, according to (2), we yield:

$$c_n(y,h) = y^{\langle n,h \rangle} \qquad (n \in \mathbb{N}_0). \quad \Box$$

Remark 5.1 Notice that in expressions (5) and (12) the deformations of variable x appear, but, contrary to that in expansions (15) and (17), the deformations of the powers of y are present.

6 Differential properties of deformed exponential functions

In this section we will look for differential operators which have deformed exponential functions as eigenfunctions.

In [5], the deformed h–differential and h–derivative were defined accordingly to operation (14):

$$\begin{aligned} d^h z &= \lim_{u \to z} z \ominus^h u, \\ \frac{df(z)}{d^h z} &= \lim_{u \to z} \frac{f(z) - f(u)}{z \ominus^h u} = \sqrt{1 + h^2 z^2}\, \frac{df(z)}{dz} . \end{aligned}$$

In this sense we can define deformed h–differential and h–derivative accordingly with operation (7) (see [4]):

$$d_h z = \lim_{u\to z} z \ominus_h u,$$
$$\frac{df(z)}{d_h z} = \lim_{u\to z} \frac{f(z)-f(u)}{z\ominus_h u} = (1+hz)\frac{df(z)}{dz}\ .$$

Theorem 6.1 *The function $x \mapsto e_h(x,y)$ is an eigenfunction of operator $\dfrac{d}{d_h x}$ with eigenvalue y.*

Proof. Let us apply differential operator $\dfrac{\partial}{\partial x}$ to function $e_h(x,y)$. Firstly, we have

$$\frac{\partial}{\partial x} e_h(x,y) = y(1+hx)^{y/h-1} = \frac{y}{1+hx}(1+hx)^{y/h} = \frac{y}{1+hx} e_h(x,y),$$

wherefrom we obtain

$$(1+hx)\frac{\partial}{\partial x}\ e_h(x,y) = y\ e_h(x,y),$$

i.e.,

$$\frac{d}{d_h x}\ e_h(x,y) = y\ e_h(x,y). \quad \square$$

Theorem 6.2 *The function $x \mapsto \exp_h(x,y)$ is an eigenfunction of operator $\dfrac{d}{d^h x}$ with eigenvalue y.*

Proof. If we apply differential operator $\dfrac{\partial}{\partial x}$ on function $\exp_h(x,y)$, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\exp_h(x,y) &= \frac{y}{h}\big(hx+\sqrt{1+h^2x^2}\big)^{\frac{y}{h}-1}\Big(h+\frac{h^2x}{\sqrt{1+h^2x^2}}\Big)\\ &= \frac{y}{\sqrt{1+h^2x^2}}\big(hx+\sqrt{1+h^2x^2}\big)^{\frac{y}{h}}\\ &= \frac{y}{\sqrt{1+h^2x^2}}\ \exp_h(x,y),\end{aligned}$$

i.e.,

$$\Big(\sqrt{1+h^2x^2}\ \frac{\partial}{\partial x}\Big)\exp_h(x,y) = y\exp_h(x,y).$$

Hence,

$$\frac{d}{d^h x}\ \exp_h(x,y) = y\ \exp_h(x,y). \quad \square$$

Finally, let us consider the behavior of deformed exponential functions related to differentiation over the second variable. In certain sense, we can conclude that they are "deformed" eigenfunctions of operator $\partial/\partial y$. Namely, the following is valid:

$$\frac{\partial}{\partial y} e_h(x,y) = \{x\}_h\ e_h(x,y), \qquad \frac{\partial}{\partial y}\exp_h(x,y) = \{x\}^h\ \exp_h(x,y).$$

7 Applications

In this section, we will note the presence and potential of deformed exponentials in growth models in the frameworks of population dynamics and compound interest in economy.

Firstly, in population dynamics, let us consider the number $N(t)$ of population individuals at the time t with initial value $N(0) = N_0$. The model assumes that the increment of population in time period δt is proportional to $N(t)$, i.e. the next difference equation is satisfied

$$\Delta_{t,\delta t} N(t) = rN(t), \tag{18}$$

where r is called the *intrinsic growth rate*. According to the Theorem 5.2, the function $t \mapsto e_{\delta t}(r,t)$ is an eigenfunction of difference operator $\Delta_{t,\delta t}$ with eigenvalue r. Hence, the solution of equation (18) can be expressed by the deformed exponential function in the form

$$N(t) = N_0 e_{\delta t}(r,t) = N_0(1 + r\delta t)^{t/\delta t}. \tag{19}$$

When $\delta t \to 0$, we get the Malthus model in population dynamics described by equation

$$\frac{d}{dt} N(t) = rN(t), \qquad N(0) = N_0,$$

whose solution is $N(t) = N_0 e^{rt}$.

In [18] and [19], the equation

$$\frac{d}{dt} t \frac{d}{dt} N(t) = rN(t), \qquad N(0) = N_0,\ N'(0) = N_1 = rN_0,$$

which describes the L–Malthus model, is discussed. In this case, the population growth increases according to the function $N(t) = N_0 e_1(rt)$, where

$$e_k(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^{k+1}} \qquad (k = 0, 1, \ldots).$$

Thus, the relevant increase is slower with respect to the classical Malthus model.

Consider a deformed h–Malthus model described by the equation

$$\frac{d}{d_h t} N(t) = rN(t), \qquad N(0) = N_0.$$

According to Theorem 6.1, its solution is

$$N(t) = N_0 e_h(t, r) = N_0(1 + ht)^{r/h}.$$

With an appropriate choice of the constant $h > 0$, we can obtain an arbitrary level of the population growth increase. Similarly, the second deformed h–Malthus model can be described by equation

$$\frac{d}{d^h t} N(t) = rN(t), \qquad N(0) = N_0,$$

and its solution

$$N(t) = N_0 \exp_h(t, r) = N_0 \left(ht + \sqrt{1 + h^2t^2}\right)^{r/h}.$$

Example 7.1 Comparison of the functions which appear in the Malthus, L–Malthus, and h–Malthus models with $r = 0.022$, $N_0 = 3346 * 10^9$ and $h = 0.003(0.003)0.024$ is shown on Figure 3. Here, the upper bold-emphasized function is $N(t) = N_0 e^{rt}$ and the lower is $N(t) = N_0 e_1(rt)$.

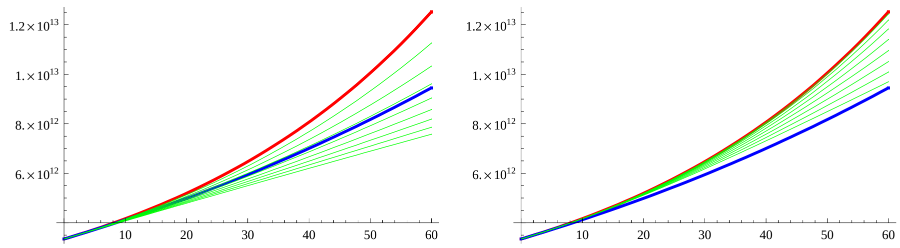

Figure 3: The Malthus, L–Malthus and h–Malthus models:
a) $N(t) = N_0 e_h(t, r)$; b) $N(t) = N_0 \exp_h(t, r)$.

Using the same model as in (18) in the area of economy, we can express the well-known law of compound interest by the deformed exponential function as

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = Pe_{1/n}(r, t),$$

where $A(t)$ is the final amount, P is the principal amount (initial investment), r is the annual nominal interest rate, n is the number of times the interest is compounded per year and t is the number of years.

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