# BEHIND THE ZEROS OF LAGUERRE POLYNOMIALS

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## 1 Introduction

The generalized Laguerre polynomials are given by

$$L_n^{(a)}(x) := \frac{1}{n!\, x^a\, e^{-x}} \frac{d^n}{dx^n}\Big(x^{a+n}\, e^{-x}\Big). \tag{1}$$

They can be also expressed by the summation formula

$$L_n^{(a)}(x) := \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-x)^k}{k!} \qquad (a > -1,\ n \in \mathbb{N}), \tag{2}$$

or

$$L_n^{(a)}(x) := \frac{\Gamma(a+n+1)}{\Gamma(n+1)} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(a+k+1)} \binom{n}{k} x^k \quad (a > -1,\ n \in \mathbb{N}). \tag{3}$$

The leading coefficient is $(-1)^n/n!$. Their generating function is

$$\sum_{k=0}^{\infty} L_k^{(a)}(x) t^k = \frac{1}{(1-t)^{a+1} e^{xt/(t-1)}}.$$

The asymptotic behavior of these polynomials is discussed in lot of books and papers (for example, see [1]).

In the paper [3], W. Koepf and D. Schmersau have made the conjecture about behavior of the generalized Laguerre polynomials $L_n^{(1)}(x)$ in the point $x = 4n + 4$ which is behind their zeros.

**Conjecture 1.1.** *For the the generalized Laguerre polynomials* $L_n^{(a)}(x)$*, it is valid*

$$\lim_{n\to\infty} \frac{-L_n^{(a)}(4n+4)}{L_{n-1}^{(a)}(4n)} \;=\; e^2 \qquad (0 < a < 6)\,. \tag{4}$$

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Knowing that for any positive sequence $\{u_n\}$ is true:

$$\lim_{n\to\infty} \frac{u_n}{u_{n-1}} = \lim_{n\to\infty} \sqrt[n]{u_n} \ , \tag{5}$$

if both limits exist, we can write conjecture in the next equivalent form.

**Conjecture 1.2.** *For the the generalized Laguerre polynomials $L_n^{(a)}(x)$, it is valid*

$$\lim_{n\to\infty} \sqrt[n]{|L_n^{(a)}(4n+4)|} = \ e^2 \qquad (0 < a < 6) \ . \tag{6}$$

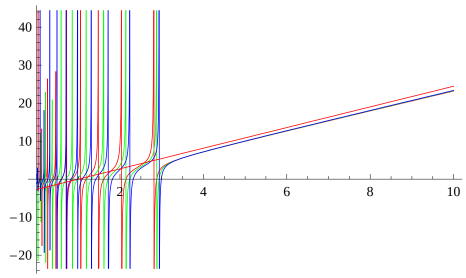


**Figure 1.** Behavior of $-L_n^{(a)}((n+1)x)/L_{n-1}^{(a)}(nx)$ for $n = 10, 11, 12$

# 2 Bounds and asymptotic relations

**Theorem 2.1.** *It is valid*

$$\lim_{n\to\infty} \sqrt[n]{|L_n^{(a)}((n+1)x)|} \le e^{x/2} \ . \tag{7}$$

*Proof.* In the paper [8], P.G.Rooney has proved

$$|L_n^{(a)}(x)| < \frac{(a+1)_n}{n!} \ e^{x/2} \qquad (a > 0, x > 0) \ . \tag{8}$$

Hence

$$\sqrt[n]{|L_n^{(a)}((n+1)x)|} < \sqrt[n]{\frac{(a+1)_n}{n!}} \ e^{\frac{n+1}{n}\frac{x}{2}} \ . \tag{9}$$

Taking limit we get the statement. ♢

From this theorem, it is obvious

$$\lim_{n\to\infty} \sqrt[n]{|L_n^{(a)}(4(n+1))|} \le e^2 \ . \tag{10}$$

**Remark.** In the paper [6], I. Krasikov has considered orthonormal Laguerre polynomials

$$\mathbf{L}_n^{(a)}(x) := \frac{L_n^{(a)}(x)}{\sqrt{n!\,\Gamma(n+a+1)}} \qquad (a > -1, x > 0).$$

He proved that it is valid

$$10^{-8} < n^{1/6}(n+a+1)^{-1/2} \max_{x\geq 0}\Big(\big(\mathbf{L}_n^{(a)}(x)\big)^2 e^{-x}\, x^{a+1}\Big) < 1444 \quad (a \geq 24;\ n \geq 35).$$

From the upper bound, we can conclude that

$$\lim_{n\to\infty} \sqrt[n]{|\mathbf{L}_n^{(a)}((n+1)x)|} \leq e^{x/2} \quad (a \geq 24)\ . \tag{11}$$

Let us denote by

$$f_n^{(a)}(x) = \frac{-L_n^{(a)}((n+1)x)}{L_{n-1}^{(a)}(nx)} \tag{12}$$

**Lemma 2.2.** *The function $f_n^{(a)}(x)$ has asymptotic line*

$$f_n^{(a)}(x) \ \sim\ \left(1+\frac{1}{n}\right)^n \left(x\ -1-\frac{a-1}{n(n+1)}\right), \quad x \to +\infty. \tag{13}$$

*Proof.* It follows from

$$\lim_{x\to\infty} \frac{f_n^{(a)}(x)}{x} \ =\ \left(1+\frac{1}{n}\right)^n, \tag{14}$$

and

$$\lim_{x\to\infty}\left(f_n^{(a)}(x) - \left(1+\frac{1}{n}\right)^n x\right) \ =\ -\left(1+\frac{1}{n}\right)^n\left(1+\frac{a-1}{n(n+1)}\right) \qquad (n\in\mathbb{N})\ .\diamondsuit \tag{15}$$

**Corollary 2.3.** *The function $f_\infty^{(a)}(x)$ has asymptotic line $y = ex - e$, i.e.*

$$f_\infty^{(a)}(x)\ \sim\ e\,x\ -\ e, \quad x\to +\infty. \tag{16}$$

# 3 The usage of binomial transform

The direct binomial transform $\mathbf{B}$ connects two sequences in the next manner:

$$\mathbf{B}:\ \{a_n\}\mapsto\{b_n\}, \qquad b_n = \mathbf{B}[\{a_0,\dots,a_n\}] = \sum_{k=0}^{n}\binom{n}{k}a_k. \tag{17}$$

Vice versa, the inverse binomial transform connect is given by sequences in the next manner:

$$\mathbf{B^{-1}}:\ \{b_n\}\mapsto\{a_n\}, \qquad a_n = \mathbf{B^{-1}}[\{b_0,\dots,b_n\}] = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}b_k. \tag{18}$$

**Lemma 3.1.** *It is valid*

$$\frac{x^n}{n!} - \frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k^{(1)}(x) \ . \tag{19}$$

*Proof.* It follows by mathematical induction. ♢

**Lemma 3.2.** *It is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} L_k^{(1)}((n+1)x)}{\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} L_k^{(1)}(nx)} = e\ x.$$

*Proof.* By (19), for $x \rightarrow (n+1)x$, we have

$$\frac{(n+1)^n x^n}{n!} - \frac{(n+1)^{n-1}x^{n-1}}{(n-1)!} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k^{(1)}((n+1)x).$$

Similarly, for $n \rightarrow n-1$ and $x \rightarrow nx$, we have

$$\frac{n^{n-1}x^{n-1}}{(n-1)!} - \frac{n^{n-2}x^{n-2}}{(n-2)!} = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k L_k^{(1)}(nx).$$

The quotient of last two formulas and limit process, verify the statement. ♢

E.D. Rainville, Special Functions, Macmillan, New York, 1960.

$$L_n^{(\alpha)}(ax) = \sum_{k=0}^{n} (\alpha+k+1)_{n-k} a^k (1-a)^{n-k} L_k^{(\alpha)}(x).$$

**Conjecture 3.3.** *It is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} L_k^{(1)}(4(n+1))}{\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} L_k^{(1)}(4n)} = e.$$

This conjecture is equivalent to

**Conjecture 3.4.** *It is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1}(-1)^k \binom{n-1}{k-1} L_k^{(1)}(4n)}{\sum_{k=0}^{n-1}(-1)^k \binom{n-1}{k} L_k^{(1)}(4n)} = 3.$$

**Lemma 3.5.** *It is valid*

$$\frac{x^n}{\Gamma(n+a+1)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{\Gamma(k+a+1)} L_k^{(a)}(x) \ . \tag{20}$$

*Proof.* The definition of Laguerre polynomials (3) we can consider like the binomial transform with

$$b_n = \frac{n!}{\Gamma(n+a+1)} L_n^{(a)}(x), \qquad a_k = \frac{(-1)^k}{\Gamma(a+k+1)}\, x^k.$$

By applying the inverse binomial transform (17), we get the statement. $\diamondsuit$

**Lemma 3.6.** *It is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(k+1)}{\Gamma(k+a+1)}L_k^{(a)}((n+1)x)}{\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{\Gamma(k+1)}{\Gamma(k+a+1)}L_k^{(a)}(nx)} = e\,x.$$

*Proof.* It follows directly from the qoutient of the formula (19) applied for $n$ with $x \to (n+1)x$ and $n-1$ with $x \to nx$ . $\diamondsuit$

**Lemma 3.7.** *It is true*

$$\lim_{n\to\infty} \frac{u_n}{u_{n-1}} = L \in \mathbb{R}\setminus\{0\} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{u_n - u_{n-1}}{u_{n-1}-u_{n-2}} = L. \tag{21}$$

*Proof.* It follows directly from

$$\lim_{n\to\infty} \frac{u_n - u_{n-1}}{u_{n-1}-u_{n-2}} = \lim_{n\to\infty} \frac{\frac{u_n}{u_{n-1}} - 1}{1 - \frac{u_{n-2}}{u_{n-1}}} = \frac{L-1}{1-L^{-1}} = L. \quad \diamondsuit$$

**Lemma 3.8.** *It is true*

$$\mathbf{B}[\{a_0,\ldots,a_n\}] - \mathbf{B}[\{a_0,\ldots,a_{n-1}\}] = \mathbf{B}[\{a_1,\ldots,a_n\}]. \tag{22}$$

*Proof.* By using formula (17), we obtain

$$\begin{aligned}
\mathbf{B}[\{a_0,\ldots,a_n\}] - \mathbf{B}[\{a_0,\ldots,a_{n-1}\}] &= \sum_{k=0}^{n}\binom{n}{k}a_k - \sum_{k=0}^{n-1}\binom{n-1}{k}a_k \\
&= a_n + \sum_{k=1}^{n-1}\left(\binom{n}{k} - \binom{n-1}{k}\right)a_k \\
&= a_n + \sum_{k=1}^{n-1}\binom{n-1}{k-1}a_k \\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}a_{i+1} \\
&= \mathbf{B}[\{a_1,\ldots,a_n\}]. \quad \diamondsuit
\end{aligned}$$

**Lemma 3.9.** *It is true*

$$\lim_{n\to\infty} \frac{\mathbf{B}[\{a_0,\ldots,a_n\}]}{\mathbf{B}[\{a_0,\ldots,a_{n-1}\}]} = 1 + \lim_{n\to\infty}\frac{a_n}{a_{n-1}}. \tag{23}$$

*Proof.* Applying (21) for $u_n = \mathbf{B}[\{a_0, \ldots, a_n\}]$ and (22), we obtain

$$\lim_{n\to\infty} \frac{\mathbf{B}[\{a_0, \ldots, a_n\}]}{\mathbf{B}[\{a_0, \ldots, a_{n-1}\}]} = \lim_{n\to\infty} \frac{\mathbf{B}[\{a_1, \ldots, a_n\}]}{\mathbf{B}[\{a_1, \ldots, a_{n-1}\}]} .$$

Repeating it $n$ times, we finally have

$$\lim_{n\to\infty} \frac{\mathbf{B}[\{a_0, \ldots, a_n\}]}{\mathbf{B}[\{a_0, \ldots, a_{n-1}\}]} = \lim_{n\to\infty} \frac{\mathbf{B}[\{a_{n-1}, a_n\}]}{\mathbf{B}[\{a_{n-1}\}]} = \lim_{n\to\infty} \frac{a_{n-1} + a_n}{a_{n-1}} ,$$

wherefrom conclusion follows. ♢

Let us denote by

$$\Phi_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k^{(a)}((k+1)x). \tag{24}$$

**Conjecture 3.10.** *It is valid*

$$\Phi_{2n}^{(a)}(x) = x^n Q_n(x), \quad \Phi_{2n-1}^{(a)}(x) = x^{n-1} R_n(x), \tag{25}$$

*where* $Q_n(x)$ *and* $R_n(x)$ *have only positive zeros.*

By previous lemma, the main conjecture (4) is equivalent with the next one.

**Conjecture 3.11.** *It is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_k^{(a)}(4(k+1))}{\sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k L_k^{(a)}(4(k+1))} = e^2 + 1 . \tag{26}$$

# 4 The shifted Laguerre polynomials

Let us denote by

$$P_n^{(a)}(x) = L_n^{(a)}\big((n+1)x\big).$$

These polynomials satisfy the next differential equation

$$x \frac{d^2}{dx^2} \tilde{P}_n^{(a)}(x) + \big(a + 1 - (n+1)x\big) \frac{d}{dx} \tilde{P}_n^{(a)}(x) + n(n+1) \tilde{P}_n^{(a)}(x) = 0.$$

They are not orthogonal in the standard sense and do not satisfy the classical three-term recurrence relation.

**Example 1.** The first three members of the sequence $\{P_n^{(1)}(x)\}$ are

$$P_1^{(1)}(x) = x - 1, \qquad P_2^{(1)}(x) = x^2 - 2x + \frac{2}{3}, \qquad P_3^{(1)} = x^3 - 3x^2 + \frac{9}{4}x - \frac{3}{8},$$

and obviously

$$P_3^{(1)} - (x-1) P_2^{(1)}(x) = \frac{7 - 10x}{24} \neq P_1^{(1)}(x).$$

We can expand $P_n^{(a)}(x)$ over the corresponding Laguerre polynomials $L_k^{(a)}(x)$ in the next way

$$P_n^{(a)}(x) = \sum_{k=0}^{n} c_{n,k} L_k^{(a)}(x),$$

where

$$c_{n,n} = (n+1)^n, \quad c_{n,k} = -\frac{n(a+k+1)}{(n+1)(n-k)} c_{n,k+1} \qquad (k = n-1, n-2, \ldots, 0).$$

These coefficients can be also written in the form

$$c_{n,n} = (n+1)^n, \qquad c_{n,k} = \frac{(-1)^{n-k}}{(n-k)!} n^{n-k} (n+1)^k \prod_{i=k}^{n-1} (a+1+i) \quad (0 \le k \le n-1).$$

**Example 2.** Here are the expansions a few $P_n^{(1)}(x)$ over the corresponding Laguerre polynomials

$$P_3^{(1)}(x) = 64L_3^{(1)}(x) - 192L_2^{(1)}(x) + 216L_1^{(1)}(x) - 108L_0^{(1)}(x),$$
$$P_4^{(1)}(x) = 625L_4^{(1)}(x) - 2500L_3^{(1)}(x) + 4000L_2^{(1)}(x) - 3200L_1^{(1)}(x) + 1280L_0^{(1)}(x).$$

# 5 Additional conjectures

Let us denote by

$$s_n = (-1)^n L_n^{(1)}(4n+4), \quad f_n = \frac{s_n}{s_{n-1}} \qquad (n \in \mathbb{N}).$$

**Conjecture 5.1.** *The sequence* $\{L_n^{(a)}(4n+4)\}$ *has the next property:*

$(i)$

$$\left(1+\frac{1}{n}\right)^{2n^2-n} < s_n < \left(1+\frac{1}{n}\right)^{2n^2+n} \qquad (n \in \mathbb{N}) \,. \tag{27}$$

$(ii)$

$$e^{2n-2} < s_n < e^{2n+1} \qquad (n \in \mathbb{N}) \,. \tag{28}$$

**Conjecture 5.2.** *The sequence* $\{f_n\}$ *has the next properties:*

$(i)$ $\{f_n\}$ *is increasing and bounded;*

$(ii)$ $\{f_n\}$ *satisfies the next inequalities:*

$$f_{n+1} < f_{n+2} < \frac{f_{n+1} + f_n}{1.85} \qquad (n \in \mathbb{N}); \tag{29}$$

$(iii)$ $\{f_n\}$ *satisfies the next inequalities:*

$$\left(1+\frac{1}{n}\right)^{2n} < f_n < \left(1+\frac{1}{n}\right)^{2n+1} \qquad (n \in \mathbb{N}); \tag{30}$$

$(iv)$ $\{f_n\}$ *is converging to* $e^2 \approx 7.38906\ldots$*.*

**Conjecture 5.3.** *For the the generalized Laguerre polynomials $L_n^{(1)}(x)$, it is valid*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} |L_k^{(1)}(4n+4)|^m}{\sum_{k=0}^{n-1} |L_k^{(1)}(4n)|^m} = e^{2m} \quad (m \in \mathbb{N}) . \tag{31}$$

*Also,*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \left(L_k^{(1)}(4n+4)\right)^m}{\sum_{k=0}^{n-1} \left(L_k^{(1)}(4n)\right)^m} = (-1)^m \, e^{2m} \quad (m \in \mathbb{N}) . \tag{32}$$

From orthogonality relation

$$\int_0^\infty t^k L_n^{(a)}(t) t^a e^{-t} dt = 0, \quad k = 0, 1, \dots, n-1,$$

by introducing new variable $x = (n+1)t$, we get

$$\int_0^\infty x^k P_n^{(a)}(x) \frac{x^a e^{-x}}{e^{nx}} dx = 0, \quad k = 0, 1, \dots, n-1.$$

**Conjecture 5.4.** *The zeros of every two different polynomials from the sequence $\{P_n^{(a)}(x)\}$ interlace each other.*

Let us denote by

$$D_n^{(a)}(x) = (-1)^n L_n^{(a)}\big((n+1)x\big) - \Big(1 + \frac{1}{n}\Big)^{2n} (-1)^{n-1} L_{n-1}^{(a)}\big(nx\big).$$

and

$$U_n^{(a)}(x) = \Big(1 + \frac{1}{n}\Big)^{2n+1} (-1)^{n-1} L_{n-1}^{(a)}\big(nx\big) - (-1)^n L_n^{(a)}\big((n+1)x\big).$$

It is valid

$$\int_0^\infty x^k \Big((-1)^n L_n^{(a)}\big((n+1)x\big) - \big(1+\frac{1}{n}\big)^{2n} (-1)^{n-1} L_{n-1}^{(a)}\big(nx\big) e^x\Big) \frac{x^a e^{-x}}{e^{nx}} dx = 0, \quad k = 0, \dots, n-2.$$

**Conjecture 5.5.** *The zeros of every two different polynomials from the polynomial sequence $\{D_n^{(a)}(x)\}$ $(0 \le a \le 6)$ are contained in $(0, 4)$ and interlace each other.*

**Conjecture 5.6.** *The zeros of every two different polynomials from the sequence the polynomial $\{U_n^{(a)}(x)\}$ $(0 \le a \le 6)$ are contained in $(0, 4)$ and interlace each other.*

*Remark* 5.1. The polynomials $\{D_n^{(a)}(x)\}$ do not satisfy three-term recurrence relation. The same is valid for $\{U_n^{(a)}(x)\}$.

## 6 G. Szegö

**Theorem 6.1.** *(G. Szegö [10]) Let $f:[0,x_1]\to\mathbb{R}$ be a convex function. Then it is valid:*

$$x_1\geq x_2\geq\cdots\geq x_{2n+1}\geq 0\quad\Rightarrow\quad f\Big(\sum_{k=1}^{2n+1}(-1)^{k-1}x_k\Big)\leq\sum_{k=1}^{2n+1}(-1)^{k-1}f(x_k)\quad(n\in\mathbb{N}).$$

**Corollary 6.2.** *It is valid:*

$$h^{n/2}\leq\sum_{k=0}^{n}(-1)^{n-k}h^k\quad(h>1;\ n\in\mathbb{N}).$$

Proof. The function $f(x)=h^x$ is a convex function for $h>1$. It is enough to take the sequence

$$n>n-1>\cdots 2>1>0=0=\cdots=0.$$

How to prove

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}4^k(n+1)^k}{k!(k+1)!(n-k)!}>\frac{1}{(n+1)!}\cdot\frac{(n+1)^{n(2n+1)}}{n^{n(2n+1)}}\ .$$

The function

$$f(x)=\frac{4^x(n+1)^x}{\Gamma(x+1)\Gamma(x+2)\Gamma(n+1-x)}$$

is not convex on $(0,n)$, but it is on $(0,n/2)$.

**Conjecture 6.3.** *It is valid*

$$\sum_{k=0}^{[n/2]}\frac{(-1)^{n-k}4^k(n+1)^k}{k!(k+1)!(n-k)!}>\frac{4^{n/4}(n+1)^{n/4}}{[n/2]!^2([n/2]+1)!}\ .$$

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