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*MATHEMATIQUES*
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# ON THE FRACTIONAL $q$–DERIVATIVE OF CAPUTO TYPE*

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**Abstract:** Based on the notion of fractional $q$–integral with the parametric lower limit of integration, we define fractional $q$–derivatives of Caputo and Riemann–Liouville type. We study some of their properties as well as the relations between them. Also, the compositions of these operators are considered.

**Key words and phrases:** basic hypergeometric functions, fractional calculus, $q$–integral, $q$–derivative, fractional integral, fractional derivative

**Mathematics Subject Classification:** 33D60, 26A33

## 1 Introduction

The fractional differential equations based on the Caputo fractional derivative require initial conditions for integer order derivatives. This is an important advantage in comparison to the approach based on Riemann-Liouville fractional derivatives in the starting point. Such type of equations are used in describing various phenomena in the science, especially in physics, chemistry, control theory and material science, because of their ability to describe memory effects, see for example [7, 4].

The discrete versions of continuous type problems in science can be made from the point of view of the so-called $q$–calculus, [3, 2, 1]. The Caputo $q$–fractional derivative has been introduced on the base of the fractional $q$–integral and fractional $q$–derivative, always with the lower limit of integration equal to 0. However, in some considerations, such as solving of $q$–differential equation of fractional order with initial values at a nonzero point, it is of interest to allow that the lower limit of integration is variable. In our papers [8] and [10], we have succeeded to generalize the theory, especially in that direction.

As a continuation, our purpose is to define the fractional $q$–derivative of Caputo type based on the fractional $q$–integrals with the parametric lower limit of integration.

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## 2 Preliminaries

In the theory of the $q$–calculus (see [5]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, a $q$–real number $[a]_q$ is introduced as

$$[a]_q := \frac{1-q^a}{1-q} \qquad (a \in \mathbb{R}) \ .$$

The $q$–analog of the Pochhammer symbol (the $q$–shifted factorial) is defined by:

$$(a;q)_0 = 1, \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad (\alpha \in \mathbb{R}) \ , \tag{1}$$

and the $q$–gamma function – by

$$\Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x} \qquad \big(x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}\big) \ . \tag{2}$$

Obviously, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The $q$–exponential functions (see [5]) can be written as power series or, applying the $q$–form of the Taylor theorem (see [8]), by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = e_q(a) \sum_{n=0}^{\infty} \frac{x^n (a/x;q)_n}{(q;q)_n} \qquad (|x|<1) \ , \tag{3}$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n = E_q(a) \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(-a;q)_n} \frac{x^n (a/x;q)_n}{(q;q)_n} \ . \tag{4}$$

The $q$–derivative of a function $f(x)$ is defined by

$$\big(D_q f\big)(x) = \frac{f(x)-f(qx)}{x-qx} \quad (x \neq 0) \ , \quad \big(D_q f\big)(0) = \lim_{x \to 0} \big(D_q f\big)(x) \ ,$$

and the $q$–derivatives of higher order:

$$D_q^0 f = f \ , \qquad D_q^n f = D_q\big(D_q^{n-1} f\big) \quad (n = 1,2,3,\ldots) \ . \tag{5}$$

The $q$-integral is defined by

$$\big(I_{q,0} f\big)(x) = \int_0^x f(t)\, d_q t = x(1-q) \sum_{k=0}^{\infty} f(xq^k)\, q^k \quad (0 \le |q| < 1), \tag{6}$$

and

$$\big(I_{q,a} f\big)(x) = \int_a^x f(t)\, d_q t = \int_0^x f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{7}$$

In the case of $q$-derivative, we can define the operator $I_{q,a}^n$ as

$$I_{q,a}^0 f = f, \qquad I_{q,a}^n f = I_{q,a}\big(I_{q,a}^{n-1} f\big) \quad (n = 1,2,3,\ldots) \ .$$

For the $q$–integral and $q$–derivative operators the following relations are valid:

$$\big(D_q^n I_{q,a}^n f\big)(x) = f(x) \qquad (n \in \mathbb{N}) \ , \tag{8}$$

$$\big(I_{q,a}^n D_q^n f\big)(x) = f(x) - \sum_{k=0}^{n-1} \frac{\big(D_q^k f\big)(a)}{[k]_q!} \ x^k (a/x;q)_k \quad (n \in \mathbb{N}) \ . \tag{9}$$

In all further considerations, we assume that the functions are defined in an interval $(0,b)$ $(b>0)$, and $a \in (0,b)$ is an arbitrary fixed point. Also, the required $q$–derivatives and $q$–integrals exist and the convergence of the series mentioned in the proofs is assumed.

**Definition 1.** The *fractional $q$–integral of order $\alpha$* is

$$\big(I_{q,a}^\alpha f\big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} \ f(t) \ d_q t \qquad (a < x; \ \alpha \in \mathbb{R}^+) \ . \tag{10}$$

The fractional $q$–integral (10) can be written in the equivalent form

$$\big(I_{q,a}^\alpha f\big)(x) = \int_a^x \ f(t) \ d_q w_\alpha(x,t) \qquad (\alpha \in \mathbb{R}^+) \ , \tag{11}$$

where $w_\alpha(x,t)$ is the function defined by

$$w_\alpha(x,t) = \frac{1}{\Gamma_q(\alpha+1)} \big(x^\alpha - x^\alpha (t/x;q)_\alpha\big) \qquad (\alpha \in \mathbb{R}^+) \ . \tag{12}$$

For any $\alpha \in \mathbb{R}^+$, the following relation is true (see [8]):

$$\big(I_{q,a}^\alpha f\big)(x) = \big(I_{q,a}^{\alpha+1} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} x^\alpha (a/x;q)_\alpha \qquad (a < x) \ . \tag{13}$$

The $q$–fractional integration has the following semigroup property (see [8]):

$$\big(I_{q,a}^\beta I_{q,a}^\alpha f\big)(x) = \big(I_{q,a}^{\alpha+\beta} f\big)(x) \qquad (a < x; \ \alpha, \beta \in \mathbb{R}^+) \ . \tag{14}$$

For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $\lambda \in (-1,\infty)$, we have the following formula:

$$I_{q,a}^\alpha \big(x^\lambda (a/x;q)_\lambda\big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+1+\alpha)} \ x^{\lambda+\alpha} (a/x;q)_{\lambda+\alpha} \qquad (a < x) \ . \tag{15}$$

On the basis of the fractional $q$–integral, we can define the $q$–derivative of real order in the following way.

**Definition 2.** The *fractional $q$–derivative of order $\alpha \in \mathbb{R}^+$ in the Riemann–Liouville sense* is understood as

$$\big(D_{q,a}^\alpha f\big)(x) = \Big(D_q^{\lceil \alpha \rceil} I_{q,a}^{\lceil \alpha \rceil - \alpha} f\Big)(x), \tag{16}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to $\alpha$. For $\alpha \le 0$, it reduces to the fractional $q$–integral, i.e. $\big(D_{q,a}^\alpha f\big)(x) = \big(I_{q,a}^{-\alpha} f\big)(x)$.

Notice that $\big(D^{\alpha}_{q,a}f\big)(x)$ has subscript $a$ to emphasize that it depends on the lower limit of integration used in definition (16). Since $\lceil\alpha\rceil$ is a positive integer for $\alpha\in\mathbb{R}^+$, then for $\big(D_q^{\lceil\alpha\rceil}f\big)(x)$ we apply definition (5).

For $a<x$ and $\alpha\in\mathbb{R}^+\setminus\mathbb{N}$, the following identity holds (see [10]):

$$\begin{aligned}\big(D_qD^{\alpha}_{q,a}f\big)(x) &= \big(D^{\alpha+1}_{q,a}f\big)(x)\ , &(17)\\ \big(D^{\alpha}_{q,a}D_qf\big)(x) &= \big(D^{\alpha+1}_{q,a}f\big)(x)-\frac{f(a)}{\Gamma_q(-\alpha)}x^{-\alpha-1}(a/x;q)_{-\alpha-1}\ . &(18)\end{aligned}$$

Also, the fractional $q$-derivative in Riemann-Liouville sense is the left and the right inverse of fractional $q$-integral, i.e.,

$$\big(I^{\alpha}_{q,a}D^{\alpha}_{q,a}f\big)(x)=\big(D^{\alpha}_{q,a}I^{\alpha}_{q,a}f\big)(x)=f(x)\qquad (a<x;\ \alpha\in\mathbb{R}^+\setminus\mathbb{N})\ . \tag{19}$$

**Remark.** Since the semigroup property is not valid for the fractional $q$–derivatives, Mansour [6] defined the sequential fractional $q$–derivative

$$\mathcal{D}^{\alpha}_q f=D^{\alpha}_{q,0}f,\qquad \mathcal{D}^{k\alpha}_q f=D^{\alpha}_{q,0}\mathcal{D}^{(k-1)\alpha}_q f,\quad k=2,3,\ldots.$$

and investigated the sequential fractional $q$–differential equations.

# 3 The fractional $q$–derivative in Caputo sense

Beside already introduced types of fractional $q$–derivatives, we will define one more. Like in standard fractional calculus, we will introduce fractional $q$–derivative of Caputo type. This one is very suitable for problems with initial values for derivatives of integer order.

**Definition 3.** The *Caputo fractional q-derivative of order* $\alpha\in\mathbb{R}^+$ is defined as

$$\big({}_{\star}D^{\alpha}_{q,a}f\big)(x)=\Big(I^{\lceil\alpha\rceil-\alpha}_{q,a}D_q^{\lceil\alpha\rceil}f\Big)(x), \tag{20}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to $\alpha$. For $\alpha\le 0$, it reduces to the fractional $q$–integral, i.e. $\big({}_{\star}D^{\alpha}_{q,a}f\big)(x)=\big(I^{-\alpha}_{q,a}f\big)(x)$.

**Example 1.** According to (15), for $\lambda>\lceil\alpha\rceil-1$, we have

$$\begin{aligned}{}_{\star}D^{\alpha}_{q,a}\big(x^{\lambda}(a/x;q)_{\lambda}\big) &= I^{\lceil\alpha\rceil-\alpha}_{q,a}D_q^{\lceil\alpha\rceil}\big(x^{\lambda}(a/x;q)_{\lambda}\big)\\ &= I^{\lceil\alpha\rceil-\alpha}_{q,a}\Big(\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\lceil\alpha\rceil+1)}x^{\lambda-\lceil\alpha\rceil}(a/x;q)_{\lambda-\lceil\alpha\rceil}\Big)\\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\lceil\alpha\rceil+1)}\ \frac{\Gamma_q(\lambda-\lceil\alpha\rceil+1)}{\Gamma_q(\lambda-\alpha+1)}\ x^{\lambda-\alpha}(a/x;q)_{\lambda-\alpha}\\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}x^{\lambda-\alpha}(a/x;q)_{\lambda-\alpha}.\end{aligned}$$

Notice that for $n\in\mathbb{N}$ and $\alpha>n$ the following holds:

$${}_{\star}D^{\alpha}_{q,a}\big(x^{n}(a/x;q)_{n}\big)=I^{\lceil\alpha\rceil-\alpha}_{q,a}D_q^{\lceil\alpha\rceil}\big(x^{n}(a/x;q)_{n}\big)=0\ .$$

**Example 2.** For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $0 < a < x < 1$, the following is valid:

$$
\begin{aligned}
{}_{\star}D^{\alpha}_{q,a}(e_q(x)) &= (1-q)^{-\alpha}\ e_q(a) \sum_{n=\lceil\alpha\rceil}^{\infty} \frac{x^{n-\alpha}(a/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}}\ , \\
{}_{\star}D^{\alpha}_{q,a}\big(E_q(x)\big) &= \frac{E_q(a)}{(1-q)^{\alpha}} \sum_{n=\lceil\alpha\rceil}^{\infty} \frac{q^{\binom{n}{2}}}{(-a;q)_n}\ \frac{x^{n-\alpha}(a/x;q)_{n-\alpha}}{(q;q)_{n-\alpha}}\ .
\end{aligned}
$$

**Theorem 1** *For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$, the following relations are valid:*

$$
\begin{aligned}
&\big({}_{\star}D^{\alpha}_{q,a} D_q f\big)(x) = \big({}_{\star}D^{\alpha+1}_{q,a} f\big)(x), \\
&\big(D_q\ {}_{\star}D^{\alpha}_{q,a} f\big)(x) = \big({}_{\star}D^{\alpha+1}_{q,a} f\big)(x) + \frac{\big(D_q^{\lceil\alpha\rceil} f\big)(a)}{\Gamma_q(\lceil\alpha\rceil-\alpha)}\ x^{\lceil\alpha\rceil-\alpha-1}(a/x;q)_{\lceil\alpha\rceil-\alpha-1}\ .
\end{aligned}
$$

*Proof.* If $\alpha = n+\varepsilon$, $n \in \mathbb{N}_0$, $0 < \varepsilon < 1$, then $\lceil\alpha\rceil = n+1$, $\lceil\alpha+1\rceil = n+2$. The first equality is valid because

$$
\begin{aligned}
({}_{\star}D^{\alpha+1}_{q,a} f)(x) &= \big(I^{\lceil\alpha+1\rceil-(\alpha+1)}_{q,a} D_q^{\lceil\alpha+1\rceil} f\big)(x) = \big(I^{1-\varepsilon}_{q,a} D_q^{n+2} f\big)(x) \\
&= \big(I^{1-\varepsilon}_{q,a} D_q^{n+1} D_q f\big)(x) = \big(I^{\lceil\alpha\rceil-\alpha}_{q,a} D_q^{\lceil\alpha\rceil} D_q f\big)(x) \\
&= ({}_{\star}D^{\alpha}_{q,a} D_q f)(x)\ .
\end{aligned}
$$

For the second equality, according to (13), we have

$$
\begin{aligned}
\big(D_q\ {}_{\star}D^{\alpha}_{q,a} f\big)(x) &= \big(D_q I^{1-\varepsilon}_{q,a} D_q^{n+1} f\big)(x) \\
&= \big(D_q I^{2-\varepsilon}_{q,a} D_q^{n+2} f\big)(x) + \frac{\big(D_q^{n+1} f\big)(a)}{\Gamma_q(2-\varepsilon)} D_q\Big(x^{1-\varepsilon}(a/x;q)_{1-\varepsilon}\Big) \\
&= \big({}_{\star}D^{\alpha+1}_{q,a} f\big)(x) + \frac{D_q^{n+1} f(a)}{\Gamma_q(n+1-\alpha)} x^{n-\alpha}(a/x;q)_{n-\alpha}\ . \ \square
\end{aligned}
$$

**Remark.** From the previous theorem, we conclude that the semigroup property for fractional $q$–derivative of Caputo type is not valid, i.e., in general

$$
\big({}_{\star}D^{\alpha}_{q,a}{}_{\star}D^{\beta}_{q,a} f\big)(x) \neq \big({}_{\star}D^{\alpha+\beta}_{q,a} f\big)(x)\ .
$$

# 4 The relations between fractional $q$–operators

The connection between the two types of the fractional $q$–derivatives is very important and it is established in the next theorem.

**Theorem 2** *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. The connection between the Caputo type and the Riemann-Liouville type fractional derivatives is as follows:*

$$
\big(D^{\alpha}_{q,a} f\big)(x) = \big({}_{\star}D^{\alpha}_{q,a} f\big)(x) + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(D_q^k f)(a)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha}(a/x;q)_{k-\alpha}
$$

*Proof.* Any $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ can be written in the form $\alpha = n + \varepsilon$, where $\varepsilon \in (0,\ 1)$. We will prove the statement by mathematical induction over $n \in \mathbb{N}_0$.

At first, let $n = 0$, i.e., $\alpha \in (0, 1)$. According to (13), we have

$$\begin{aligned}\big(I_{q,a}^{1-\alpha} f\big)(x) &= \big(I_{q,a}^{2-\alpha} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(2-\alpha)} x^{1-\alpha}(a/x;q)_{1-\alpha} \\ &= \Big(I_{q,a}\big({}_\star D_{q,a}^{\alpha} f\big)\Big)(x) + \frac{f(a)}{\Gamma_q(2-\alpha)} x^{1-\alpha}(a/x;q)_{1-\alpha}\ .\end{aligned}$$

By $q$–deriving, we get

$$\big(D_q I_{q,a}^{1-\alpha} f\big)(x) = \Big(D_q I_{q,a}\big({}_\star D_{q,a}^{\alpha} f\big)\Big)(x) + \frac{f(a)}{\Gamma_q(2-\alpha)} D_q\Big(x^{1-\alpha}(a/x;q)_{1-\alpha}\Big),$$

and, finally,

$$\big(D_q^{\alpha} f\big)(x) = \big({}_\star D_{q,a}^{\alpha} f\big)(x) + \frac{f(a)}{\Gamma_q(1-\alpha)} x^{-\alpha}(a/x;q)_{-\alpha}\ .$$

Suppose that the statement is valid for a real $\alpha = n+\varepsilon$, $\varepsilon \in (0,1)$, for a positive integer $n \in \mathbb{N}$ and let us prove that it is valid for $\alpha = n+1+\varepsilon$. Indeed, according to (17), the next equality is valid:

$$\big(D_{q,a}^{\alpha} f\big)(x) = \Big(D_q D_{q,a}^{n+\varepsilon} f\Big)(x).$$

With respect to the inductional assumption

$$\big(D_{q,a}^{n+\varepsilon} f\big)(x) = \big({}_\star D_{q,a}^{n+\varepsilon} f\big)(x) + \sum_{k=0}^{n} \frac{(D_q^k f)(a)}{\Gamma_q(1+k-n-\varepsilon)} x^{k-n-\varepsilon}(a/x;q)_{k-n-\varepsilon},$$

we can write

$$\begin{aligned}&\big(D_{q,a}^{\alpha} f\big)(x) \\ &\quad= \big(D_q\ {}_\star D_{q,a}^{n+\varepsilon} f\big)(x) + \sum_{k=0}^{n} \frac{(D_q^k f)(a)}{\Gamma_q(1+k-n-\varepsilon)} D_q\big(x^{k-n-\varepsilon}(a/x;q)_{k-n-\varepsilon}\big) \\ &\quad= \big(D_q\ {}_\star D_{q,a}^{n+\varepsilon} f\big)(x) + \sum_{k=0}^{n} \frac{(D_q^k f)(a)}{\Gamma_q(k-n-\varepsilon)}\ x^{k-n-1-\varepsilon}(a/x;q)_{k-n-1-\varepsilon}\ .\end{aligned}$$

Using Theorem 1, we obtain

$$\big(D_q\ {}_\star D_{q,a}^{n+\varepsilon} f\big)(x) = \big({}_\star D_{q,a}^{n+1+\varepsilon} f\big)(x) + \frac{\big(D_q^{n+1} f\big)(a)}{\Gamma_q(1-\varepsilon)}\ x^{-\varepsilon}(a/x;q)_{-\varepsilon}\ .$$

Finally,

$$\begin{aligned}\big(D_{q,a}^{\alpha} f\big)(x) &= \big({}_\star D_{q,a}^{n+1+\varepsilon} f\big)(x) + \frac{\big(D_q^{n+1} f\big)(a)}{\Gamma_q(1-\varepsilon)}\ x^{-\varepsilon}(a/x;q)_{-\varepsilon} \\ &\quad+ \sum_{k=0}^{n} \frac{(D_q^k f)(a)}{\Gamma_q(k-n-\varepsilon)}\ x^{k-n-1-\varepsilon}(a/x;q)_{k-n-1-\varepsilon} \\ &= \big({}_\star D_{q,a}^{\alpha} f\big)(x) + \sum_{k=0}^{n+1} \frac{(D_q^k f)(a)}{\Gamma_q(k-n-\varepsilon)}\ x^{k-n-1-\varepsilon}(a/x;q)_{k-n-1-\varepsilon}\ .\square\end{aligned}$$

Here, we will discuss the behavior of compositions of the previously defined operators.

**Theorem 3** *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, for $a < x$, the following is valid:*

$$\left(I_{q,a}^{\alpha}\ {}_{\star}D_{q,a}^{\alpha} f\right)(x) = f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f\right)(a)}{[k]_q!}\ x^k (a/x;q)_k\ .$$

*Proof.* With respect to the property (14) and the formulas (8) and (9), we have

$$\begin{aligned}\left(I_{q,a}^{\alpha}\ {}_{\star}D_{q,a}^{\alpha} f\right)(x) &= \left(I_{q,a}^{\alpha} I_{q,a}^{\lceil\alpha\rceil-\alpha} D_q^{\lceil\alpha\rceil} f\right)(x) = \left(I_{q,a}^{\lceil\alpha\rceil} D_q^{\lceil\alpha\rceil} f\right)(x)\\ &= f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f\right)(a)}{[k]_q!}\ x^k (a/x;q)_k\ . \quad \square\end{aligned}$$

**Theorem 4** *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $a < x$. Then the next relation is true:*

$$\left({}_{\star}D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) = f(x)\ .$$

*Proof.* Putting $f \mapsto I_{q,a}^{\alpha} f$ into Theorem 2, and using (19), we get

$$\begin{aligned}\left({}_{\star}D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) &= \left(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\right)(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(D_q^k I_{q,a}^{\alpha} f)(a)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha} (a/x;q)_{k-\alpha}\\ &= f(x) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{(I_{q,a}^{\alpha-k} f)(a)}{\Gamma_q(1+k-\alpha)} x^{k-\alpha} (a/x;q)_{k-\alpha} = f(x)\ . \quad \square\end{aligned}$$

In similar way, by using Theorem 2 and relation (19), the next properties can be proven.

**Theorem 5** *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ and $\beta \in \mathbb{R}^+$. Then, for $a < x$, the following is valid:*

$$\begin{aligned}&\left({}_{\star}D_{q,a}^{\alpha} I_{q,a}^{\beta} f\right)(x)\\ &\quad = \left({}_{\star}D_{q,a}^{\alpha-\beta} f\right)(x) + \sum_{k=0}^{\lceil\alpha-\beta\rceil-1} \frac{\left(D_q^k f\right)(a)}{\Gamma_q(k-\alpha+\beta+1)} x^{k-\alpha+\beta} (a/x;q)_{k-\alpha+\beta}\ ,\\ &\left(I_{q,a}^{\beta}\ {}_{\star}D_{q,a}^{\alpha} f\right)(x)\\ &\quad = \left({}_{\star}D_{q,a}^{\alpha-\beta} f\right)(x) - \sum_{k=\lceil\alpha-\beta\rceil}^{\lceil\alpha\rceil-1} \frac{\left(D_q^k f\right)(a)}{\Gamma_q(k-\alpha+\beta+1)} x^{k-\alpha+\beta} (a/x;q)_{k-\alpha+\beta}\ .\end{aligned}$$

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