THE DEFORMED AND MODIFIED MITTAG–LEFFLER POLYNOMIALS *

Miomir S. Stanković
Faculty of Ocupational Safety

Sladjana D. Marinković
Faculty of Electronic Engineering

Predrag M. Rajković
Faculty of Mechanical Engineering

University of Niš, Serbia

Abstract. The starting point of this paper are the Mittag–Leffler polynomials investigated in details by H. Bateman in [1]. Based on generalized integer powers of real numbers and deformed exponential function, we introduce deformed Mittag–Leffler polynomials defined by appropriate generating function. We investigate their recurrence relations, hypergeometric representation and orthogonality. Since they have all zeros on imaginary axes, we also consider real polynomials with real zeros associated to them.

Mathematics Subject Classification (2010): 33C45, 11B83
Key Words: Generating function, Polynomial sequence, Recurrence relation, Orthogonality

1 Introduction

The development in various disciplines of modern physics leads to including some deformed and generalized versions of the exponential function (see [10] and [5]). In this sense, we have defined the deformed exponential function of two variables in [9]. Its properties make it very suitable in theoretical considerations and applications. In this paper, we will present some classes of polynomials, related to Mittag–Leffler polynomials, based on this function.

Let us remind that the Mittag–Leffler polynomials $\{g_n(y)\}$ are the coefficients in expansion

$$\left(\frac{1+x}{1-x}\right)^y = \sum_{n=0}^{\infty} g_n(y)x^n \qquad (|x| < 1).$$

*Supported by Ministry of Sci. & Tech. of Serbia, the projects No. 174011 and 44006.

They were introduced by Mittag-Leffler in a study on the integral representations. Their main properties were found by H. Bateman (see [1] and [2]). Although they are known for a long time, a few new papers considering them, appeared recently (see, for example, [3] and [7]).

The Mittag–Leffler polynomials $\{g_n(y)\}$ can be represented by hypergeometric function like

$$g_n(y) = 2y \,{}_2F_1\left(\begin{matrix} 1-n, \; 1-y \\ 2 \end{matrix} \;\middle|\; 2 \right) \qquad (n \in \mathbb{N}). \tag{1}$$

In that sense, they can be considered as a special case of the Meixner polynomials $M_n(x;\beta,c)$ for $\beta = 2$ and $c = -1$ (see [6]). Namely, their relation is

$$g_n(y) = 2yM_{n-1}(y-1, 2, -1).$$

The lack of this connection is the fact that the Meixner polynomials are mostly used with constraint $0 < c < 1$ which guarantees discrete orthogonality and other good properties .

Also, the Mittag–Leffler polynomials $g_n(y)$ are connected with the Pidduck polynomials [11] by the expression $P_n(y) = ((e^D + 1)/2)g_n(y)$, where we use series for the exponential function and D is understood as differentiation.

The Mittag–Leffler polynomials $\{g_n(y)\}$ satisfy the recurrence relations

$$g_n(y+1) - g_{n-1}(y+1) = g_n(y) + g_{n-1}(y),$$

$$(n+1)g_{n+1}(y) - 2yg_n(y) + (n-1)g_{n-1}(y) = 0,$$

both with initial values

$$g_0(y) = 1, \quad g_1(y) = 2y,$$

and orthogonality relation

$$\int_{-\infty}^{+\infty} g_n(-iy)g_m(iy)\frac{dy}{y\sinh \pi y} = \frac{2}{n}\delta_{mn} \qquad (n, m \in \mathbb{N}), \tag{2}$$

where δ_{mn} is the Kronecker delta.

Notice that the corresponding monic sequence

$$\hat{g}_n(y) = \frac{(n)!}{2^n}\, g_n(y) \qquad (n \in \mathbb{N}_0)$$

has the exponential generating function

$$\left(\frac{2+x}{2-x}\right)^y = \sum_{n=0}^{\infty} \hat{g}_n(y)\frac{x^n}{n!} \;.$$

2 A few generalizations of well-known notions

Let $h \in \mathbb{R} \setminus \{0\}$. We define generalized integer powers of real numbers [8], as

$$z^{(0,h)} = z^{[0,h]} = 1, \quad z^{(n,h)} = \prod_{k=0}^{n-1} (z - kh), \quad z^{[n,h]} = \prod_{k=0}^{n-1} (z + kh) \quad (n \in \mathbb{N}).$$

Recall the h–difference operator is (see [4])

$$\Delta_{z,h} f(z) = \frac{f(z+h) - f(z)}{h} = \frac{1}{h}\big(E_h - I\big) f(z), \tag{3}$$

where I is the identity and E_h is the shift–operator. The h–difference operator is linear with the next product rule:

$$\Delta_{z,h}\big(f(z)g(z)\big) = f(z+h)\Delta_{z,h} g(z) + \Delta_{z,h} f(z) g(z). \tag{4}$$

Its action on integer generalized powers is given by:

$$\Delta_{z,h}\, z^{(n,h)} = n z^{(n-1,h)}, \qquad \Delta_{z,h}\, z^{[n,h]} = n(z+h)^{[n-1,h]}.$$

In [9], it was defined the deformed exponential function

$$e_h(x,y) = (1+hx)^{y/h} \qquad (x \in \mathbb{C} \setminus \{-1/h\},\ y \in \mathbb{R}).$$

The function $e_h(x,y)$ keeps some of the basic properties of the exponential function. For $y \in \mathbb{R}$, the following holds:

$$\begin{aligned} & e_h(x,y) > 0 \qquad (x < -1/h \quad \text{for } h < 0 \quad \text{or} \quad x > -1/h \quad \text{for } h > 0), \\ & e_h(0,y) = e_h(x,0) = 1. \end{aligned}$$

If h changes the sign, we have

$$e_{-h}(x,y) = e_h(-x,-y) \qquad (x \neq 1/h). \tag{5}$$

The additive property is kept only in regard to the second variable:

$$e_h(x,y_1) e_h(x,y_2) = e_h(x, y_1 + y_2).$$

The deformed exponential functions can be represented as the expansions:

$$e_h(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n y^{(n,h)} \qquad (|hx| < 1), \tag{6}$$

$$e_{-h}(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n y^{[n,h]} \qquad (|hx| < 1). \tag{7}$$

If we apply h–difference operators on the deformed exponential functions, we get:

$$\Delta_{y,h}\ e_h(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!}\, x^n n y^{(n-1,h)} = x\ e_h(x,y), \tag{8}$$

$$\Delta_{y,h}\ e_{-h}(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!}\, x^n n (y+h)^{[n-1,h]} = x\ e_{-h}(x, y+h). \tag{9}$$

An interesting differential property of this function (see [9]) is given as

$$\left((1+hx)\frac{\partial}{\partial x}\right)\ e_h(x,y) = y\ e_h(x,y). \tag{10}$$

3 The deformed Mittag–Leffler polynomials

By usage of previously defined function, the generating function of Mittag–Leffler polynomials can be rewritten as

$$G(x,y) = (1+x)^y(1-x)^{-y} = e_1(x,y)\ e_{-1}(x,y).$$

For $h \in \mathbb{R} \setminus \{0\}$ we can define *the deformed Mittag–Leffler polynomials* as the coefficients in expansion

$$G_h(x,y) = e_h(x,y)\ e_{-h}(x,y) = \sum_{n=0}^{\infty} g_n^{(h)}(y)x^n\ . \tag{11}$$

With respect to (6) and (7), we have

$$\begin{aligned} e_h(x,y)e_{-h}(x,y) &= \sum_{n=0}^{\infty}\frac{y^{(n,h)}}{n!}x^n \sum_{m=0}^{\infty}\frac{y^{[m,h]}}{m!}x^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{y^{(m,h)}y^{[n-m,h]}}{m!(n-m)!}x^n. \end{aligned}$$

Hence,

$$g_n^{(h)}(y) = \sum_{m=0}^{n}\frac{y^{(m,h)}y^{[n-m,h]}}{m!(n-m)!} = \frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}y^{(m,h)}y^{[n-m,h]} \qquad (n \in \mathbb{N}_0).$$

According to (5), we have

$$G_h(x,y) = G_h(-x,-y) = \sum_{n=0}^{\infty} g_n^{(h)}(-y)(-1)^n x^n,$$

and, consequently,

$$g_n^{(h)}(-y) = (-1)^n g_n^{(h)}(y).$$

Also, from $G_h(x,y) = G_{-h}(x,y)$ we get

$$g_n^{(-h)}(y) = g_n^{(h)}(y).$$

Let us derive some recurrence relations for the polynomials $g_n^{(h)}(y)$.

Theorem 3.1 *The successive members of sequence $\{g_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ satisfy the three–term recurrence relation*

$$\begin{aligned} (n+1)g_{n+1}^{(h)}(y) - 2yg_n^{(h)}(y) - h^2(n-1)g_{n-1}^{(h)}(y) &= 0 \quad (n \geq 2), \\ g_0^{(h)}(y) = 1, \qquad g_1^{(h)}(y) &= 2y. \end{aligned}$$

Proof. Firstly, we have

$$\frac{\partial}{\partial x}G_h(x,y)=\left(\frac{\partial}{\partial x}e_h(x,y)\right)e_{-h}(x,y)+e_h(x,y)\left(\frac{\partial}{\partial x}e_{-h}(x,y)\right).$$

Multiplying by $(1-hx)(1+hx)$, with respect to the differential property (10) of $e_h(x,y)$, we obtain:

$$\begin{aligned}(1-h^2x^2)&\frac{\partial}{\partial x}G_h(x,y)\\ &=(1-hx)ye_h(x,y)e_{-h}(x,y)+(1+hx)e_h(x,y)ye_{-h}(x,y)\\ &=2yG_h(x,y).\end{aligned}$$

Using (11) and comparing the coefficients in the series we obtain the recurrence relation. □

Example 3.1 The first members of the sequence $\{g_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ are:

$$g_0^{(h)}(y)=1,\quad g_1^{(h)}(y)=2y,\quad g_2^{(h)}(y)=2y^2,\quad g_3^{(h)}(y)=\frac{2}{3}y(2y^2+h^2),$$

$$g_4^{(h)}(y)=\frac{2}{3}y^2(y^2+2h^2),\quad g_5^{(h)}(y)=\frac{2}{15}y(2y^4+10h^2y^2+3h^4).$$

Especially, let us emphasize the relation

$$\frac{\partial}{\partial x}G_h(x,y)=\frac{2y}{1-h^2x^2}G_h(x,y) \tag{12}$$

obtained in the proof of Theorem 3.1, which will be useful in further research.

Theorem 3.2 *For the polynomials $g_n^{(h)}(y)$ the following relation is valid:*

$$g_n^{(h)}(y+h)-g_n^{(h)}(y)=h\big(g_{n-1}^{(h)}(y+h)+g_{n-1}^{(h)}(y)\big)\qquad(n\in\mathbb{N}).$$

Proof. According to (4), (8) and (9) we have

$$\begin{aligned}\Delta_{y,h}G_h(x,y)&=\Delta_{y,h}\big(e_h(x,y)e_{-h}(x,y)\big)\\ &=e_h(x,y+h)\Delta_{y,h}e_{-h}(x,y)+e_{-h}(x,y)\Delta_{y,h}e_h(x,y)\\ &=x\big(e_h(x,y+h)e_{-h}(x,y+h)+e_h(x,y)e_{-h}(x,y)\big)\\ &=x\big(G_h(x,y+h)+G_h(x,y)\big).\end{aligned}$$

Using (11), because of the linearity of operator $\Delta_{y,h}$ the last equation becomes

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_{y,h}g_n^{(h)}(y)x^n&=x\sum_{n=0}^{\infty}\big(g_n^{(h)}(y+h)+g_n^{(h)}(y)\big)x^n\\ &=\sum_{n=1}^{\infty}\big(g_{n-1}^{(h)}(y+h)+g_{n-1}^{(h)}(y)\big)x^n.\end{aligned}$$

With respect to (3), we obtain the required relation. □

Theorem 3.3 *The polynomial $g_n^{(h)}(y)$ can be represented by hypergeometric function as*

$$g_n^{(h)}(y) = 2yh^{n-1}\ {}_2F_1\Big(\begin{matrix}1-n,\ 1-y/h\\ 2\end{matrix}\ \Big|\ 2\Big) \qquad (n \in \mathbb{N}). \tag{13}$$

Proof. It follows from the previous theorem by mathematical induction. Namely, its equivalent form is summation formula

$$\begin{aligned} g_n^{(h)}(y) &= 2yh^{n-1} \sum_{k=0}^{n-1} \frac{(1-n)_k(1-y/h)_k}{(2)_k}\frac{2^k}{k!} \\ &= 2yh^{n-1} \sum_{k=0}^{n-1}(1-y/h)_k \frac{(-1)^k\ 2^k}{k!(k+1)!}\prod_{j=1}^{k}(n-j). \end{aligned}$$

It is true for $n = 1$. We will suppose that it is valid for all $k \le n$. Then

$$\begin{aligned} &(n+1)g_{n+1}^{(h)}(y) - 2yg_n^{(h)}(y) - h^2(n-1)g_{n-1}^{(h)}(y) \\ &\qquad = 2yh^n\Big((n+1)\sum_{k=0}^{n}(1-y/h)_k\frac{(-1)^k\ 2^k}{k!(k+1)!}\prod_{j=1}^{k}(n+1-j) \\ &\qquad\quad + 2\sum_{k=0}^{n-1}(1-y/h)_k(-y/h)\frac{(-1)^k\ 2^k}{k!(k+1)!}\prod_{j=1}^{k}(n-j) \\ &\qquad\quad - (n-1)\sum_{k=0}^{n-2}(1-y/h)_k\ \frac{(-1)^k\ 2^k}{k!(k+1)!}\prod_{j=1}^{k}(n-1-j)\Big). \end{aligned}$$

Since

$$(1-y/h)_{k+1} = (1-y/h)_k(-y/h) + (k+1)(1-y/h)_k,$$

the coefficient of the term $(1-y/h)_k$ $(0 \le k \le n)$ in the recurrence relation is

$$\frac{(-1)^k 2^k}{k!(k+1)!}\Big((n+1)n-(n-k)(n-k-1)-k(k+1)-2(k+1)(n-k)\Big)\prod_{j=1}^{k-1}(n-j) = 0. \ \square$$

Let us show that the polynomials of the sequence $\{g_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ are orthogonal.

Theorem 3.4 *For the members of the sequence $\{g_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ the following orthogonality relation is valid:*

$$\int_{-\infty}^{\infty} g_n^{(h)}(iy)g_m^{(h)}(-iy)\frac{dy}{y\sinh(\pi y/h)} = \frac{2h^{2n-2}}{n}\ \delta_{mn} \qquad (m, n \in \mathbb{N}). \tag{14}$$

Proof. Using representation (1) and the integral representation of the hypergeometric function, we can write

$$\begin{aligned} g_n^{(h)}(y) &= 2yh^{n-1}\frac{\Gamma(2)}{\Gamma(1-y/h)\Gamma(1+y/h)}\int_0^1 u^{-y/h}(1-u)^{y/h}(1-2u)^{n-1}du \\ &= \frac{h^n}{\pi}\sin\frac{\pi y}{h}\int_{-1}^{1} u^{n-1}(1+u)^{y/h}(1-u)^{-y/h}du, \end{aligned}$$

or, by changing of variable $u = \tanh(ht/2)$,

$$g_n^{(h)}(y) = \frac{h^n}{\pi} \sin \frac{\pi y}{h} \int_{-\infty}^{\infty} \Big(\tanh \frac{ht}{2}\Big)^n \frac{e^{ty}}{\sinh ht}\, dt.$$

Hence,

$$\begin{aligned} g_n^{(h)}(iy) &= \frac{ih^n}{\pi} \sinh \frac{\pi y}{h} \int_{-\infty}^{\infty} e^{ity} \Big(\tanh \frac{ht}{2}\Big)^n \frac{dt}{\sinh ht} \\ &= ih^n \sqrt{\frac{2}{\pi}}\, \sinh \frac{\pi y}{h}\, \mathcal{F}\left(\Big(\tanh \frac{ht}{2}\Big)^n \frac{1}{\sinh ht}\right), \end{aligned}$$

where $\varphi(t) \mapsto \Phi(y) = \mathcal{F}(\varphi(t))$ denotes the Fourier transform. Applying the inverse Fourier transform, we get

$$\Big(\tanh \frac{ht}{2}\Big)^n \frac{1}{\sinh ht} = \frac{1}{ih^n} \sqrt{\frac{\pi}{2}}\, \mathcal{F}^{-1}\left(\frac{g_n^{(h)}(iy)}{\sinh(\pi y/h)}\right),$$

i.e.,

$$\Big(\tanh \frac{ht}{2}\Big)^n = \frac{1}{2ih^n} \sinh ht \int_{-\infty}^{\infty} e^{-ity} \frac{g_n^{(h)}(iy)}{\sinh(\pi y/h)}\, dy. \tag{15}$$

Further, according to (12), we recognize

$$\begin{aligned} \sinh ht\ e^{-ity} &= \frac{2\tanh(ht/2)}{1-\tanh^2(ht/2)} \left(\frac{1+\tanh(ht/2)}{1-\tanh(ht/2)}\right)^{-iy/h} \\ &= \frac{ih}{y} \tanh \frac{ht}{2}\ \frac{\partial}{\partial x} G_h(x,-iy)\, \Big|_{x=\frac{1}{h}\tanh(ht/2)}. \end{aligned}$$

Because of

$$\frac{\partial}{\partial x} G_h(x,-iy)\, \Big|_{x=\frac{1}{h}\tanh(ht/2)} = \sum_{m=1}^{\infty} \frac{m}{h^{m-1}} g_m^{(h)}(-iy) \left(\tanh \frac{ht}{2}\right)^{m-1},$$

we obtain

$$\sinh ht\ e^{-ity} = \frac{ih}{y} \sum_{m=1}^{\infty} \frac{m}{h^{m-1}} g_m^{(h)}(-iy) \left(\tanh \frac{ht}{2}\right)^{m}.$$

Substituting last equation in (15), we have

$$\begin{aligned} &\Big(\tanh \frac{ht}{2}\Big)^n \\ &= \frac{1}{2h^{n-1}} \sum_{m=1}^{\infty} \frac{m}{h^{m-1}} \left(\tanh \frac{ht}{2}\right)^{m} \int_{-\infty}^{\infty} \frac{g_n^{(h)}(iy) g_m^{(h)}(-iy)}{y \sinh(\pi y/h)}\, dy. \end{aligned}$$

Comparing the coefficients by $\tanh(ht/2)$, we get required relation. □

The monic sequence $\{\hat{g}_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ is related with $\{g_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ by

$$g_n^{(h)}(y) = \frac{2^n}{n!}\hat{g}_n^{(h)}(y).$$

They satisfy the three–term recurrence relation

$$\begin{aligned} &\hat{g}_{n+1}^{(h)}(y) = y\,\hat{g}_n^{(h)}(y) + h^2\frac{n(n-1)}{4}\,\hat{g}_{n-1}^{(h)}(y) \qquad (n\in\mathbb{N}),\\ &\hat{g}_0^{(h)}(y) = 1, \qquad \hat{g}_1^{(h)}(y) = y, \end{aligned}$$

and have generating function given by

$$\hat{G}_h(x,y) = e_h(x/2,y)e_{-h}(x/2,y) = \sum_{n=0}^{\infty}\hat{g}_n^{(h)}(y)\frac{x^n}{n!}\ .$$

4 The modified polynomials of the Mittag-Leffler and the deformed Mittag–Leffler polynomials

Let us consider *the modified Mittag-Leffler polynomials* defined by

$$\varphi_n(y) = \frac{g_{n+1}(iy)}{i^{n+1}\ y} \qquad (n\in\mathbb{N}_0). \tag{16}$$

Theorem 4.1 *The successive members of sequence $\{\varphi_n(y)\}_{n\in\mathbb{N}_0}$ satisfy the three–term recurrence relation*

$$\begin{aligned} (n+2)\varphi_{n+1}(y) &= 2y\varphi_n(y) - n\ \varphi_{n-1}(y) \quad (n\in\mathbb{N}) \\ \varphi_0(y) &= 2, \qquad \varphi_1(y) = 2y. \end{aligned} \tag{17}$$

Proof. The stated relation follows from the recurrence relation (2). □

Theorem 4.2 *The generating function of sequence $\{\varphi_n(y)\}_{n\in\mathbb{N}_0}$ is given by*

$$\mathcal{G}(x,y) = \frac{\exp(2y\arctan x) - 1}{xy} = \sum_{n=0}^{\infty}\varphi_n(y)x^n. \tag{18}$$

Proof. Starting from recurrence relation (17) and summarizing, we have:

$$\sum_{n=1}^{\infty}(n+2)\varphi_{n+1}(y)x^n - 2y\sum_{n=1}^{\infty}\varphi_n(y)x^n + \sum_{n=1}^{\infty}n\varphi_{n-1}(y)x^n = 0,$$

$$\sum_{n=2}^{\infty}(n+1)\varphi_n(y)x^{n-1} - 2y\sum_{n=1}^{\infty}\varphi_n(y)x^n + \sum_{n=0}^{\infty}(n+1)\varphi_n(y)x^{n+1} = 0,$$

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\sum_{n=2}^{\infty}\varphi_n(y)x^n\right) - 2y\sum_{n=1}^{\infty}\varphi_n(y)x^n + x\frac{\partial}{\partial x}\left(x\sum_{n=0}^{\infty}\varphi_n(y)x^n\right) = 0,$$

i.e.,

$$(1+x^2)\frac{\partial}{\partial x}\big(x\mathcal{G}(x,y)\big) - 2yx\mathcal{G}(x,y) - 2 = 0.$$

Solving obtained differential equation with initial condition $x\mathcal{G}(x,y)\big|_{x=0} = 0$, we get generating function. □

Notice that

$$\varphi_n(-y) = (-1)^n\ \varphi_n(y) \qquad (n \in \mathbb{N}). \tag{19}$$

Theorem 4.3 *The polynomials of sequence* $\{\varphi_n(y)\}_{n\in\mathbb{N}_0}$ *satisfy the following orthogonality relation:*

$$\int_{-\infty}^{+\infty} \varphi_n(y)\varphi_m(y)\frac{y}{\sinh(\pi y)}\ dy = \frac{2}{n+1}\ \delta_{mn} \qquad (n,m \in \mathbb{N}_0).$$

Proof. Using (2) and (16), we find

$$-i^{n+m}\int_{-\infty}^{+\infty} \varphi_n(-y)\varphi_m(y)\frac{y^2}{y\sinh(\pi y)}\ dy = \frac{2}{n+1}\ \delta_{mn} \qquad (n,m \in \mathbb{N}_0),$$

what, with respect to (19), gives orthogonality relation. □

The monic sequence

$$\hat\varphi_n(y) = \frac{(n+1)!}{2^{n+1}}\ \varphi_n(y) \qquad (n \in \mathbb{N}_0)$$

satisfies three term recurrence relation

$$\begin{aligned} \hat\varphi_{n+1}(y) &= y\hat\varphi_n(y) - \frac{n(n+1)}{4}\ \hat\varphi_{n-1}(y) \quad (n \in \mathbb{N}) \\ \hat\varphi_0(y) &= 1, \qquad \hat\varphi_1(x) = y. \end{aligned} \tag{20}$$

Theorem 4.4 *The exponential generating function of sequence* $\{\hat\varphi_n(y)\}_{n\in\mathbb{N}_0}$ *is given by*

$$\hat{\mathcal{G}}(x,y) = \frac{4\exp\big(2y\ \arctan(x/2)\big)}{x^2+4} = \sum_{n=0}^{\infty}\hat\varphi_n(y)\frac{x^n}{n!}. \tag{21}$$

Proof. In order to find exponential generating function of polynomials $\hat\varphi(y)$

$$\hat{\mathcal{G}}(x,y) = \sum_{n=0}^{\infty}\hat\varphi_n(y)\frac{x^n}{n!},$$

we will start with recurrence relation (20) and summarize

$$\sum_{n=1}^{\infty}\Big(\hat\varphi_{n+1}(y) - y\hat\varphi_n(y) + \frac{n(n+1)}{4}\ \hat\varphi_{n-1}(y)\Big)\frac{x^n}{n!} = 0,$$

i.e.

$$\sum_{n=2}^{\infty}\hat\varphi_n(y)\frac{x^{n-1}}{(n-1)!} - y\sum_{n=1}^{\infty}\hat\varphi_n(y)\frac{x^n}{n!} + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)(n+2)\ \hat\varphi_n(y)\frac{x^{n+1}}{(n+1)!} = 0.$$

Hence

$$\frac{\partial}{\partial x}\hat{\mathcal{G}}(x,y) - y\hat{\mathcal{G}}(x,y) + \frac{1}{4}\frac{\partial}{\partial x}\Big(x^2\hat{\mathcal{G}}(x,y)\Big) = 0.$$

This is the simple differential equation

$$\frac{d\hat{\mathcal{G}}}{\hat{\mathcal{G}}} = 2\frac{2y-x}{x^2+4}\,dx$$

with initial value $\hat{\mathcal{G}}(0,y) = 1$. Its solution is the function (21). $\Box$

Example 4.1 The first members of the sequence $\{\hat{\varphi}(y)\}_{n\in\mathbb{N}_0}$ are

$$\hat{\varphi}_0(y) = 1, \quad \hat{\varphi}_1(y) = y, \quad \hat{\varphi}_2(y) = y^2 - \frac{1}{2}, \quad \hat{\varphi}_3(y) = y^3 - 2y,$$

$$\hat{\varphi}_4(y) = y^4 - 5y^2 + \frac{3}{2}, \quad \hat{\varphi}_5(y) = y^5 - 10y^3 + \frac{23}{2}y.$$

In the same manner, we can define *the modified deformed Mittag-Leffler polynomials* by

$$\varphi_n^{(h)}(y) = \frac{g_{n+1}^{(h)}(iy)}{i^{n+1}\;y} \qquad (n \in \mathbb{N}_0).$$

The properties of sequence $\{\varphi_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ can be derived in the same way as in case of $\{\varphi_n(y)\}_{n\in\mathbb{N}_0}$. Hence, we give the results without proofs.

Corollary 4.1 *The successive members of the sequence $\{\varphi_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ satisfy the three–term recurrence relation*

$$\begin{aligned}(n+2)\varphi_{n+1}^{(h)}(y) &= 2y\varphi_n^{(h)}(y) - h^2 n\;\varphi_{n-1}^{(h)}(y) \quad (n \in \mathbb{N})\\ \varphi_0^{(h)}(y) &= 2, \qquad \varphi_1(y)^{(h)} = 2y.\end{aligned}$$

Also, the following is valid:

$$\varphi_n^{(h)}(-y) = (-1)^n\;\varphi_n^{(h)}(y) \qquad (n \in \mathbb{N}).$$

Corollary 4.2 *The generating function of the sequence $\{\varphi_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ is given by*

$$\mathcal{G}_h(x,y) = \frac{1}{xy}\left(\exp\Big(2\frac{y}{h}\arctan hx\Big) - 1\right) = \sum_{n=0}^{\infty}\varphi_n^{(h)}(y)x^n.$$

Corollary 4.3 *The polynomials of the sequence $\{\varphi_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ satisfy following orthogonality relation:*

$$\int_{-\infty}^{+\infty}\varphi_n^{(h)}(y)\varphi_m^{(h)}(y)\frac{y}{\sinh(\pi y/h)}\;dy = \frac{2h^{2n}}{n+1}\;\delta_{mn} \qquad (n,m \in \mathbb{N}_0).$$

The monic sequence

$$\hat{\varphi}_n^{(h)}(y) = \frac{(n+1)!}{2^{n+1}}\, \varphi_n^{(h)}(y) \qquad (n \in \mathbb{N}_0)$$

satisfies three term recurrence relation

$$\begin{aligned}\hat{\varphi}_{n+1}^{(h)}(y) &= y\hat{\varphi}_n^{(h)}(y) - \frac{h^2}{4} n(n+1)\, \hat{\varphi}_{n-1}^{(h)}(y) \quad (n \in \mathbb{N}) \\ \hat{\varphi}_0^{(h)}(y) &= 1, \qquad \hat{\varphi}_1^{(h)}(x) = y.\end{aligned}$$

Corollary 4.4 *The exponential generating function of* $\{\hat{\varphi}_n^{(h)}(y)\}_{n\in\mathbb{N}_0}$ *is*

$$\hat{\mathcal{G}}_h(x,y) = \frac{4}{4+h^2x^2}\ \exp\Big(2\frac{y}{h}\arctan\frac{hx}{2}\Big) = \sum_{n=0}^{\infty} \hat{\varphi}_n^{(h)}(y)\frac{x^n}{n!}\ .$$

Example 4.2 The first members of the sequence $\{\hat{\varphi}^{(h)}(y)\}_{n\in\mathbb{N}_0}$ are

$$\hat{\varphi}_0^{(h)}(y) = 1, \quad \hat{\varphi}_1^{(h)}(y) = y, \quad \hat{\varphi}_2^{(h)}(y) = y^2 - \frac{h^2}{2}, \quad \hat{\varphi}_3^{(h)}(y) = y^3 - 2h^2y,$$

$$\hat{\varphi}_4^{(h)}(y) = y^4 - 5h^2y^2 + \frac{3}{2}h^4, \quad \hat{\varphi}_5^{(h)}(y) = y^5 - 10h^2y^3 + \frac{23}{2}h^4y.$$

The modified Mittag-Leffler $\{\varphi_n(y)\}$ and the deformed Mittag–Leffler polynomials $\{\varphi_n^{(h)}(y)\}$ are real polynomials and, because of the orthogonality, they have all real and distinct zeros.

References

[1] H. Bateman, *The polynomial of Mittag–Leffler*, Proc N.A.S. **26** (1940) 491–496.

[2] H. Bateman, *An orthogonal property of the hypergeometric polynomial*, Proc N.A.S. **28** (1942) 374–377.

[3] T.X. He, L.C. Hsu, P.J.-S. Shiue, *The Sheffer group and the Riordan group*, Discrete Applied Mathematics **155** (2007) 1895-1909.

[4] D. Jagerman, *Difference equations with applications to queues*, Marcel Dekker Inc., 2000.

[5] G. Kaniadakis, *Towards a relativistic statistical theory*, Physica A **365** (2006) 17-23.

[6] R. Koekoek, R.F. Swarttouw, *"Askey-scheme of hypergeometric orthogonal polynomials and its q–analogue"*, Report of Delft University of Technology No 98-17, 1998.

[7] A. Luzon, M.A. Moron, *Recurrence relations for polynomial sequences via Riordan matrices*, Linear Algebra and its Applications, (accepted for publishing in 2010.)

[8] J. Riordan, *Combinatorial Identities*, Reprint of the 1968 original, Robert E. Krieger Publishing Co., Huntington, NY, 1979.

[9] M.S. Stanković, S.D. Marinković, P.M. Rajković, *Deformed exponential functions of two variables*, ArXiv 1005.5040v1, May 27, 2010. http://arxiv.org/abs/1005.5040v1

[10] C. Tsallis, *Introduction to Non-extensive Statistical Mechanics*, Springer, 2009.

[11] E.W. Weisstein, *Pidduck Polynomial*,MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/PidduckPolynomial.html