On a class of almost orthogonal polynomials

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Abstract. In this paper, we define a new class of almost orthogonal polynomials which can be used successfully for modelling of electronic systems which generate orthonormal basis. Especially, for the classical weight function, they can be considered like a generalization of the classical orthogonal polynomials (Legendre, Laguerre, Hermite, ...). They are very suitable for analysis and synthesis of imperfect technical systems which are projected to generate orthogonal polynomials, but in the reality generate almost orthogonal polynomials.

Key words: Orthogonality, Polynomials, Approximation.

1 Introduction

It is known that orthogonal polynomials are useful tool in technical sciences. Here we will emphasize their role in signal approximation ([1],[2]) and design of the electronic systems which generate the orthogonal signals ([3], [4]). However, since the components of those systems can not be made quite exactly, the polynomials which are generated by these systems are not quite orthogonal, but rather *almost orthogonal*. The measure of nearness between the obtained and the regular orthogonal polynomials depends on the exactness of the manufacturing of the components. Until now, some classes of almost orthogonal functions are defined and investigated by other authors and their applications are considered too (see, for example, [5], [6] and [7]).

Let $\lambda(x)$ be a positive Borel measure on an interval $(a, b) \subset \mathbb{R}$ with infinite support and such that all moments

$$\lambda_n = \mathcal{L}[x^n] = \int_a^b x^n d\lambda(x) \tag{1}$$

exist. In this manner, we define linear functional $\mathcal{L}$ in the linear space of real polynomials $\mathcal{P}$. Also, we can introduce an inner product as follows (see [8]):

$$(f,\ g) = \mathcal{L}\big[f \cdot g\big] \qquad (f, g \in \mathcal{P}), \tag{2}$$

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which is positive-definite because of the property $\|f\|^2 = (f,f) \geq 0$. Hence it follows that monic polynomials $\{P_n(x)\}$ orthogonal with respect to this inner product exist and they satisfy the three-term recurrence relation

$$P_{k+1}(x) = (x-\alpha_k)P_k(x) - \beta_k P_{k-1}(x) \quad (k \geq 0), \qquad P_{-1} \equiv P_0 \equiv 1. \tag{3}$$

For a very small positive real number ε, let us consider a functional

$$\mathcal{L}_\varepsilon[f] = \mathcal{L}[f] - \varepsilon \qquad (0 < \varepsilon \ll 1). \tag{4}$$

If it exists, the sequence of monic polynomials $\{P_n^{(\varepsilon)}(x)\}$ which satisfies the relations

$$\mathcal{L}_\varepsilon\left[P_j^{(\varepsilon)} \cdot P_k^{(\varepsilon)}\right] = 0 \ (j \neq k), \qquad |||P_k^{(\varepsilon)}|||_\varepsilon^2 = \mathcal{L}_\varepsilon\left[\left(P_k^{(\varepsilon)}\right)^2\right] > 0 \ (j,k \in \mathbb{N}_0) \tag{5}$$

will be called *orthogonal* with respect to $\mathcal{L}_\varepsilon$, or *almost orthogonal* w.r.t. $\mathcal{L}$.

Obviously, it is valid

$$\lim_{\varepsilon \to 0} P_n^{(\varepsilon)}(x) = P_n(x). \tag{6}$$

Notice that

$$\mathcal{L}_\varepsilon[cf(x)] = c\mathcal{L}_\varepsilon[f(x)] + (c-1)\varepsilon, \qquad \mathcal{L}_\varepsilon[f(x)+g(x)] = \mathcal{L}_\varepsilon[f(x)] + \mathcal{L}_\varepsilon[g(x)] + \varepsilon,$$

wherefrom the presence of additional ε abandons the distribution property. It means that a lot of simple procedures useful for the classical orthogonal polynomials cannot be applied here.

We have to use the next procedure for their construction. Denoting

$$\mu_{j,k} = \mathcal{L}\left[x^j P_k^{(\varepsilon)}(x)\right],$$

we can find the coefficients in

$$P_n^{(\varepsilon)}(x) = x^n + \sum_{k=0}^{n-1} a_{n,k}\, x^k$$

like solutions of the linear algebraic system

$$\begin{bmatrix} \mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\ \mu_{0,1} & \mu_{1,1} & & \mu_{n-1,1} \\ \vdots & & & \\ \mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1} \end{bmatrix} \cdot \begin{bmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n-1} \end{bmatrix} = \begin{bmatrix} \varepsilon - \mu_{n,0} \\ \varepsilon - \mu_{n,1} \\ \vdots \\ \varepsilon - \mu_{n,n-1} \end{bmatrix}.$$

2 The connection between almost orthogonal and orthogonal polynomials

Since the monic polynomials $\{P_k(x)\}_{k=0}^n$ form a basis in the space of the polynomials $\mathcal{P}_n$ whose degree is not greater than n, we can write an expression as follows:

$$P_n^{(\varepsilon)}(x) = P_n(x) + \sum_{k=0}^{n-1} \frac{b_k^{(n)}(\varepsilon)}{\|P_k\|^2} P_k(x). \tag{7}$$

Theorem 1. *The coefficients* $b_j^{(n)}$ *do not depend on* n, *i.e.*

$$b_j^{(n)}(\varepsilon) = b_j(\varepsilon) \qquad (\forall n \in \mathbb{N}_0). \tag{8}$$

Proof. By taking the inner product of $P_n^{(\varepsilon)}(x)$ with P_j, we can write

$$(P_n^{(\varepsilon)}, P_j) = (P_n, P_j) + \sum_{k=0}^{n-1} \frac{b_k^{(n)}(\varepsilon)}{\|P_k\|^2}(P_k, P_j).$$

By the orthogonality relations for the polynomials $\{P_k(x)\}_{k=0}^{n}$, we get

$$(P_n^{(\varepsilon)}, P_j) = b_j^{(n)}(\varepsilon).$$

On the other hand, we can express

$$P_j(x) = \sum_{i=0}^{j} c_{j,i}(\varepsilon) P_i^{(\varepsilon)}(x).$$

Hence

$$b_j^{(n)}(\varepsilon) = \sum_{i=0}^{j} c_{j,i}(\varepsilon)(P_n^{(\varepsilon)}, P_i^{(\varepsilon)}) = \varepsilon \sum_{i=0}^{j} c_{j,i}(\varepsilon),$$

wherefrom we see that $b_j(\varepsilon)$ does not depend on n. $\diamondsuit$

Theorem 2. *The coefficients* $b_k = b_k(\varepsilon)$ $(k \in \mathbb{N}_0)$ *satisfy the next recurrence relation*

$$b_n + \frac{b_{n-1}^2}{\|P_{n-1}\|^2} + \cdots + \frac{b_0^2}{\|P_0\|^2} = \varepsilon \quad (n \in \mathbb{N}). \tag{9}$$

Proof. From the fact

$$\mathcal{L}_\varepsilon\Big(P_{n+1}^{(\varepsilon)} \cdot P_n^{(\varepsilon)}\Big) = 0 \quad \Leftrightarrow \quad \big(P_{n+1}^{(\varepsilon)}, P_n^{(\varepsilon)}\big) = \varepsilon,$$

and from the orthogonality of $\{P_k(x)\}$, we have

$$\begin{aligned}
\varepsilon &= \Big(P_{n+1}(x) + \textstyle\sum_{k=0}^{n} \frac{b_k}{\|P_k\|^2} P_k(x)\ ,\ P_n(x) + \sum_{i=0}^{n-1} \frac{b_i}{\|P_i\|^2} P_i(x)\Big) \\
&= \textstyle\sum_{k=0}^{n} \frac{b_k}{\|P_k\|^2}\Big(P_k(x)\ ,\ P_n(x) + \sum_{i=0}^{n-1} \frac{b_i}{\|P_i\|^2} P_i(x)\Big) \\
&= \textstyle\frac{b_n}{\|P_n\|^2}\big(P_n(x), P_n(x)\big) + \sum_{k=0}^{n-1} \frac{b_k}{\|P_k\|^2}\Big(P_k(x)\ ,\ \frac{b_k}{\|P_k\|^2} P_k(x)\Big),
\end{aligned}$$

wherefrom the formula follows. $\diamondsuit$

By mathematical induction and the previous theorem, we can prove the next recurrence relation.

Corollary 1. *The polynomials* $\{b_k(\varepsilon)\}$ $(\deg b_k(\varepsilon) = 2^k)$ *are related recurrently by*

$$b_0 = \varepsilon, \quad b_{k+1} = b_k\Big(1 - \frac{b_k}{\|P_k\|^2}\Big) \qquad (k \in \mathbb{N}). \tag{10}$$

Proof. We get it by subtracting the formula from the previous theorem for $n = k+1$ and $n = k$. $\diamondsuit$

Corollary 2. *The squared norm of the polynomials* $\{P_n^{(\varepsilon)}(x)\}$ *is*

$$\|P_n^{(\varepsilon)}\|^2 = \|P_n\|^2 + \sum_{k=0}^{n-1} \frac{b_k^2}{\|P_k\|^2}.$$

Proof. From the definition of norm and orthogonality of polynomials, we get

$$\begin{aligned}
\|P_n^{(\varepsilon)}\|^2 &= (P_n^{(\varepsilon)}, P_n^{(\varepsilon)}) \\
&= \Big(P_n(x) + \textstyle\sum_{k=0}^{n-1} \frac{b_k}{\|P_k\|^2} P_k(x)\ ,\ P_n(x) + \sum_{i=0}^{n-1} \frac{b_i}{\|P_i\|^2} P_i(x)\Big) \\
&= (P_n, P_n) + \textstyle\sum_{k=0}^{n-1} \frac{b_k}{\|P_k\|^2} \Big(P_k(x)\ ,\ \sum_{i=0}^{n-1} \frac{b_i}{\|P_i\|^2} P_i(x)\Big) \\
&= \|P_n\|^2 + \textstyle\sum_{k=0}^{n-1} \frac{b_k}{\|P_k\|^2} \Big(P_k(x)\ ,\ \frac{b_k}{\|P_k\|^2} P_k(x)\Big),
\end{aligned}$$

wherefrom the conclusion follows. $\diamondsuit$

Remark 1. Unfortunately, the recurrence relation for $\{P_n^{(\varepsilon)}(x)\}$ is not known. Comparing the first members of the sequence, we easily conclude that they do not satisfy recurrence relation of the form (3).

3 Some classical cases

The monic *shifted Legendre polynomials* $P_n(x)$ are orthogonal with respect to the weight $w(x) \equiv 1$ on $(0, 1)$. They satisfy the three-term recurrence relation

$$P_{k+1}(x) = (x - \alpha_k)P_k(x) - \beta_k P_{k-1}(x) \quad (k \ge 0), \qquad P_{-1} \equiv P_0 \equiv 1,$$

with

$$\alpha_k = \frac{1}{2} \quad (k \ge 0), \qquad \beta_0 = 1, \quad \beta_k = \frac{k^2}{4(4k^2 - 1)} \quad (k > 0). \tag{11}$$

The square norm is

$$\|P_0\|^2 = \beta_0 = 1, \qquad \|P_n\|^2 = \prod_{k=0}^{n} \beta_k = \frac{1}{4^n} \frac{(n!)^2}{(2n-1)!!(2n+1)!!} \quad (n > 0). \tag{12}$$

and their explicit form is

$$P_n(x) = \frac{n!}{(2n)!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{(n+j)!}{j!} x^j. \tag{13}$$

Let us consider the functional

$$\mathcal{L}_\varepsilon[f] = \int_0^1 f(x)dx - \varepsilon \qquad (0 < \varepsilon \ll 1) \tag{14}$$

and the sequence of monic polynomials $\{P_n^{(\varepsilon)}(x)\}$ which satisfies the relation of orthogonality w.r.t. $\mathcal{L}_\varepsilon$.

The first members of this sequence are:

$$\left\{1, \quad x - (\frac{1}{2} - \varepsilon), \quad x^2 - (1 - 12\varepsilon + 12\varepsilon^2)x + \frac{1}{6}(1 - 30\varepsilon + 36\varepsilon^2)\right\}.$$

Now, in the expansion (7), we get the coefficients b_k as follows:

$$b_0 = \varepsilon, \qquad b_1 = (1 - \varepsilon)b_0, \qquad b_2 = (1 - 12\varepsilon + 12\varepsilon^2)b_1.$$

The monic *Chebyshev polynomials of the first kind* $T_n(x)$ are orthogonal with respect to the weight $w(x) = 1/\sqrt{1 - x^2}$ on $(-1, 1)$. They satisfy the three-term recurrence relation

$$T_{k+1}(x) = xT_k(x) - \beta_k T_{k-1}(x) \quad (k \geq 0), \qquad T_{-1} \equiv T_0 \equiv 1,$$

with

$$\beta_1 = \frac{1}{2}, \qquad \beta_k = \frac{1}{4} \quad (k \geq 2). \tag{15}$$

Also, the norms are: $\|T_0\|^2 = \pi,\ \|T_n\|^2 = \pi/2^{2n-1} \quad (n \geq 1)$. The explicit form is

$$T_n(x) = n \sum_{k=0}^{[n/2]} \frac{(-1)^k (n - k - 1)!}{k!(n - 2k)!\ 4^k} x^{n-2k}. \tag{16}$$

If we define the functional

$$\mathcal{L}_\varepsilon[f] = \int_{-1}^1 f(x) \frac{dx}{\sqrt{1 - x^2}} - \varepsilon \qquad (0 < \varepsilon \ll 1), \tag{17}$$

we can introduce the sequence of polynomials $\{T_n^{(\varepsilon)}(x)\}$ orthogonal with respect to $\mathcal{L}_\varepsilon$:

$$\left\{1, \quad x + \frac{\varepsilon}{\pi}, \quad x^2 + \frac{2\varepsilon(\pi - \varepsilon)}{\pi^2}x - \frac{\pi - 2\varepsilon}{2\pi}, \dots\right\},$$

with

$$b_0 = \varepsilon, \qquad b_1 = \frac{\pi - \varepsilon}{\pi} b_0, \qquad b_2 = \frac{\pi^2 - 2\pi\varepsilon + 2\varepsilon^2}{\pi^2}\ b_1, \dots\ .$$

The *Laguerre polynomials* $L_n(x)$ satisfy the three-term recurrence relation

$$L_{k+1}(x) = (x - \alpha_k)L_k(x) - \beta_k L_{k-1}(x) \quad (k \geq 0), \qquad L_{-1} \equiv L_0 \equiv 1,$$

with

$$\alpha_k = 2k+1 \quad (k \geq 0), \qquad \beta_0 = 1, \quad \beta_k = k^2 \quad (k > 0). \tag{18}$$

Their explicit expression and the norm are:

$$L_n(x) = n! \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{x^k}{k!}, \qquad \|L_n\| = n! \qquad (n \in \mathbb{N}) \,.$$

We will denote by $\{L_n^{(\varepsilon)}(x)\}$ the sequence of polynomials which satisfies the relation of orthogonality with respect to the functional

$$\mathcal{L}_\varepsilon[f] = \int_0^{+\infty} f(x)\ \mathrm{e}^{-x} dx - \varepsilon \qquad (0 < \varepsilon \ll 1). \tag{19}$$

The first three members of this sequence are:

$$\big\{1, \quad x - (1-\varepsilon), \quad x^2 - (4 - \varepsilon + \varepsilon^2)x + (2 + \varepsilon^2)\big\}\,.$$

Now, the expansion (7) involves the following polynomials over ε:

$$b_0 = \varepsilon, \qquad b_1 = (1-\varepsilon) b_0, \qquad b_2 = (1 - \varepsilon + \varepsilon^2) b_1.$$

4 The least square approximation over almost orthogonal polynomials

For an integrable function $f(x)$ we are considering the approximations over the polynomials $\{P_k^{(\varepsilon)}(x)\}$:

$$f(x) \approx \Phi_n(x) = \sum_{k=0}^{n} d_k P_k^{(\varepsilon)}(x) \qquad (n \in \mathbb{N}).$$

The error of the approximation can be estimated by

$$\mathcal{G}(d_0, d_1, \ldots, d_n) = \||\Phi_n(x) - f(x)\||_\varepsilon^2.$$

The least square approximation w.r.t. such introduced norm will be reached in the point of minimum of the function $\mathcal{G}(d_0, d_1, \ldots, d_n)$. Hence

$$\frac{1}{2}\ \frac{\partial \mathcal{G}}{\partial d_j} \equiv \sum_{k=0}^{n} d_k (P_k^{(\varepsilon)}, P_j^{(\varepsilon)}) - (f, P_j^{(\varepsilon)}) = 0 \qquad (0 \leq j \leq n).$$

So we get the system of linear algebraic equations

$$\|P_j^{(\varepsilon)}\|^2\ d_j + \varepsilon \sum_{k:k\neq j} d_k = \big(f, P_j^{(\varepsilon)}\big) \qquad (j = 0, 1, \ldots, n).$$

Denoting by $s_k = \|P_k^{(\varepsilon)}\|^2$ and $f_k = (f, P_k^{(\varepsilon)})$ $(0 \le k \le n)$, we can write the matrix form of this system like

$$\begin{bmatrix} s_0 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & s_1 & & \varepsilon \\ \vdots & & & \\ \varepsilon & \varepsilon & & s_n \end{bmatrix} \cdot \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}. \tag{20}$$

By Cramer's rule, we can evaluate d_k's. At first, we find that the determinant of the system is

$$\Delta(\varepsilon) = \prod_{i=0}^{n} (s_i - \varepsilon)\Big(1 + \varepsilon \sum_{i=0}^{n} \frac{1}{s_i - \varepsilon}\Big).$$

Including the Corollary 2, we see that the acceptable values for ε are:

$$0 < \varepsilon \ll \tau_n = \min_{0 \le k \le n} \|P_k\|^2 \qquad (\forall n \in \mathbb{N}_0). \tag{21}$$

If we want to evaluate coefficients in approximation we have to find determinants $\Delta_k(\varepsilon)$ which we get from $\Delta(\varepsilon)$ putting free elements in the $(k+1)$th column. Now, we have

$$\Delta_k(\varepsilon) = \sum_{i=0}^{n} u_i f_i,$$

where

$$u_k = \prod_{\substack{i=0 \\ i \ne k}}^{n} (s_i - \varepsilon)\left(1 + \varepsilon \sum_{\substack{i=0 \\ i \ne k}}^{n} \frac{1}{s_i - \varepsilon}\right), \qquad u_i = \varepsilon\, g_i(\varepsilon)\ (i \ne k),$$

with $g_i(\varepsilon)$ a polynomial over ε. It is bounded function for $\varepsilon \in (0, \tau_n)$.

Obviously, for $\varepsilon = 0$, this approximation becomes the least square approximation over the orthogonal polynomials $\{P_n(x)\}$ and the system (20) is diagonal and well-coditioned.

But, for $\varepsilon \ne 0$, we must use almost orthogonal instead of orthogonal polynomials. Although theoretically we can achieve the approximation of the same quality, in practical computing it is not always true. The system (20) can be ill-conditioned for bigger ε, that is why it is important to use condition (21).

Example 1. The sequence $\{\|P_n\|^2\}$ of squared norms of shifted Legendre polynomials is decreasing rapidly to zero. Really, for $n = 5$, the exactness of expansion of the function $f(x) = \ln(1+x)\cos 5x$ over $\{P_j^{(\varepsilon)}(x)\}$ is holding on for $\varepsilon < \|P_5\|^2 = 1.43155(-6)$, but for larger it can decrease significantly. Numerical evaluating shows that for $\varepsilon = 0.01$, the exactness is still good $1.1883(-6)$, but for $\varepsilon = 0.02$, it is only $3.6331(-4)$.

If we compare approximations on $(-1, 1)$ with Chebyshev weight and on $(0, \infty)$ with Laguerre weight, the influence of $\varepsilon \in (0, 1)$ is negligible because the norm-sequence is slowly decreasing in Chebyshev case in spite of the increase in Laguerre case.

Conjecture 1. Under condition (21), the polynomials $\{P_n^{(\varepsilon)}(x)\}$ *have all zeros on the support interval* (a,b).

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