Closed-form expression for Hankel determinants of the Narayana polynomials

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Abstract

The Hankel transform of a sequence whose elements are the Narayana polynomials is considered. We discuss its properties and express the values in the closed form.

Key words: Narayana numbers, Hankel transform, orthogonal polynomials.

Mathematics Subject Classification: 11Y55 , 34A25

1 Introduction

The *Narayana numbers* were firstly studied by P.A. MacMahon [16]. The name is given in honour to T.V. Narayana who later rediscovered them in [17].

Definition 1.1. *Narayana numbers $(N(n,k))_{n,k\in\mathbb{N}_0}$ are defined by the following relations* [15, 19, 20]:

$$\begin{aligned} N(0,k) &= N(k,0) = \delta_{k0}, \quad (k \in \mathbb{N}_0), \\ N(n,k) &= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \quad (n,k \in \mathbb{N}). \end{aligned} \tag{1}$$

Here δ_{ij} denotes Kronecker delta function. The infinite matrix whose elements are Narayana numbers looks like:

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 3 & 1 & 0 & 0 & \\ 0 & 1 & 6 & 6 & 1 & 0 & \\ 0 & 1 & 10 & 20 & 10 & 1 & \\ \vdots & & & & & & \ddots \end{bmatrix}.$$

That sequence is denoted by A001263 in the On-Line Encyclopedia of Integer Sequences [19]. The corresponding sequence of *Narayana polynomials* $(a(n;r))_{n\in\mathbb{N}_0}$ is defined by

$$a(n;r) = \sum_{k=0}^{n} N(n,k)r^k. \tag{2}$$

Example 1.1. The first few members of the sequence of Narayana polynomials are:

$$\begin{aligned}
a(0;r) &= N(0,0) = 1,\\
a(1;r) &= N(1,0) + N(1,1)r = r,\\
a(2;r) &= N(2,0) + N(2,1)r + N(2,2)r^2 = r + r^2,\\
a(3;r) &= N(3,0) + N(3,1)r + N(3,2)r^2 + N(3,3)r^3 = r + 3r^2 + r^3.
\end{aligned}$$

Narayana numbers and polynomials actuate a lot of attention due to their various combinatorial interpretations (see [1, 2, 20]).

Connection between Narayana numbers and lattice paths was studied by R.A. Sulanke in [20]. Sulanke considered lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,-1)$ to (n,n), and proved that the number of such paths of length k with step size from $\mathbb{N}\times\mathbb{N}$, such that whole path remain strictly above the line $y = x-1$, is equal to $N(n,k)$.

Furthermore, Branden considered *Dyck paths*, i.e. paths with the step size from the set $\{(0,1),(1,0)\}$ (i.e. in each step you can move up or forward by unit distance). It is proved that the number of colored Dyck paths, such that each corner of this path is colored in one of the r colors, is $a(n;r)$.

Various of other interesting properties of Narayana numbers and polynomials can be found in [2, 15, 20, 21]. Narayana polynomials are refining the famous Catalan sequence $\mathcal{C}_n$, i.e. $a(n;1) = \mathcal{C}_n$ (see for example [15, 20]). Moreover, $(a(n;2))_{n\in\mathbb{N}_0}$ is the sequence of *large Schröder numbers* [1, 3].

In order to provide an easier evaluation of the Hankel transform of Narayana polynomials, we introduce the sequences of *modified Narayana numbers and polynomials.*

Definition 1.2. *The modified Narayana numbers* $(N_1(n,k))_{n,k\in\mathbb{N}_0}$ *are defined by the following relations:*

$$N_1(0,k) = \delta_{k0}, \quad N_1(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1} \qquad (n\in\mathbb{N};\ k\in\mathbb{N}_0)$$

The infinite matrix whose elements are members of the sequence of modified Narayana numbers is given by:

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & 0 & \\ 1 & 6 & 6 & 1 & 0 & 0 & \\ 1 & 10 & 20 & 10 & 1 & 0 & \\ \vdots & & & & & & \end{bmatrix}.$$

The corresponding *modified Narayana polynomials* are

$$a_1(n;r) = \sum_{k=0}^{n-1} N_1(n,k)r^k, \tag{3}$$

while the first few terms of that sequence are given by:

Example 1.2. The first few terms of *modified Narayana polynomials* are:

$$\begin{aligned}
a_1(0;r) &= N_1(0,0) = 1,\\
a_1(1;r) &= N_1(1,0) + N_1(1,1)r = 1\\
a_1(2;r) &= N_1(2,0) + N_1(2,1)r + N_1(2,2)r^2 = 1 + r,\\
a_1(3;r) &= N_1(3,0) + N_1(3,1)r + N_1(3,2)r^2 + N_1(3,3)r^3 = 1 + 3r + r^2.
\end{aligned}$$

The connections between Narayana numbers (polynomials) and modified Narayana numbers (polynomials) are given by

$$N_1(n,k) = N(n,k+1); \qquad a(0;r) = a_1(0;r) = 1, \quad a(n;r) = r \cdot a_1(n;r) \qquad (n \in \mathbb{N}) \ . \tag{4}$$

Remark 1.1. The sequence $(a_1(n;r))_{n\in\mathbb{N}_0}$ reduces to the sequence of *Catalan numbers* for $r = 1$, and to *little Schröder numbers* for $r = 2$.

The rest of the paper is organized as follows. In the Section 2, we recall definition and basic properties of Hankel transform. Also, we introduce the method based on orthogonal polynomials for Hankel transform evaluation. That method is applied on the sequence of Narayana polynomials, in two steps. Firstly, in Section 3, we evaluate Hankel transform of the shifted Narayana polynomials. In continuation, in Section 4, we prove a few additional properties of the Hankel transform which are used for deriving the final result, together with the evaluation in the previous section. Section 5 provides the further results concerning the Hankel transform evaluation of sums of two consecutive shifted Narayana polynomials.

2 Hankel transform and series reversion

An important transform on integer sequences that has been much studied recently is the Hankel transform [5, 14, 18]:

Definition 2.1. *The* **Hankel transform** *of a given sequence* $a = (a_n)_{n\in\mathbb{N}_0}$ *is the sequence of Hankel determinants* $(h_n)_{n\in\mathbb{N}_0}$ *where* $h_n = \det[a_{i+j-2}]_{i,j=1}^n$*, i.e*

$$a = (a_n)_{n\in\mathbb{N}_0} \quad \Longrightarrow^{\mathcal{H}} \quad h = (h_n)_{n\in\mathbb{N}_0} : \quad h_n = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{bmatrix} \tag{5}$$

We denote Hankel transform by $\mathcal{H}$ *and hence we write* $h = \mathcal{H}(a)$*.*

Hankel transform was widely investigated in numerous papers. For example, the papers [5, 18] deal with the Hankel transform of the sequences of the Catalan and adjusted Catalan numbers. Both evaluations are done by using method based on orthogonal polynomials. From the other side, evaluation of Hankel determinants with orthogonal polynomial entries is investigated by M.E.H. Ismail in [10]. Many other Hankel transform evaluations are given in the papers [6, 7, 11, 22]. In this paper we also use method based on orthogonal polynomials, as used in [5, 18].

Let $(a_n)_{n\in\mathbb{N}_0}$ be the moment sequence with respect to some measure $d\lambda$. In other words, let

$$a_n = \int_{\mathbb{R}} x^n d\lambda \quad (n = 0, 1, 2, \dots) . \tag{6}$$

Then the Hankel transform $h = \mathcal{H}(a)$ of the sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed by the following relation known as Heilermann formula (for example, see Krattenthaler [12])

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n. \tag{7}$$

The sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \beta_n Q_{n-1}(x) , \tag{8}$$

where $(Q_n(x))_{n\in\mathbb{N}_0}$ is the monic polynomial sequence, orthogonal with respect to the measure $d\lambda$. The following theorem and corollary provide the way how to explicitly find the measure $d\lambda$ with prescribed moment sequence.

Theorem 2.1. (Stieltjes-Perron inversion formula) [4, 13] *Let $(\mu_n)_{n\in\mathbb{N}_0}$ is the sequence such that all elements of its Hankel transform are non-negative. Denote by $G(z) = \sum_{n=0}^{+\infty}\mu_n z^n$ the generating function of the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ and $F(z) = z^{-1}G(z^{-1})$. Also let the function $\lambda(t)$ is defined by*

$$\lambda(t) - \lambda(0) = -\frac{1}{2\pi i}\lim_{y\to 0^+}\int_0^t \Big[F(x+iy) - F(x-iy)\Big]dx. \tag{9}$$

Then holds $\mu_n = \int_{\mathbb{R}} x^n d\lambda$, i.e. sequence $(\mu_n)_{n\in\mathbb{N}_0}$ is the moment sequence of the measure $\lambda(t)$.

Corollary 2.2. *Under the assumptions of the previous lemma, let additionally holds $F(\bar{z}) = \overline{F(z)}$. Then*

$$\lambda(t) - \lambda(0) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_0^t \Im F(x+iy)dx. \tag{10}$$

We use Theorem 2.1 and Corollary 2.2 to determine weight function (measure) such that the given sequence is moment sequence to that measure. After that, by applying weight function transformations and using Heilermann formula (7), we provide closed-form expression for the Hankel transform. In order to efficiently use Theorem 2.1 and Corollary 2.2, we need expression for the weight function of our sequence.

At last, for our further discussion we need notion of series reversion (see [1]).

Definition 2.2. *For a given function $v = f(u)$ with the property $f(0) = 0$, the series reversion is the sequence $\{s_k\}_{k\in\mathbb{N}_0}$ such that*

$$u = f^{-1}(v) = s_0 + s_1 v + \cdots + s_n v^n + \cdots ,$$

where $u = f^{-1}(v)$ is the inverse function of $v = f(u)$.

3 The Hankel transform of the shifted Narayana polynomials

Define the temporary sequence of *shifted Narayana polynomials* $a_{sh}(n;r) = a_1(n+1;r) = r^{-1}a(n;r)$. That sequence is introduced since it is the moment sequence with respect to the absolutely continuous measure.

3.1 Evaluation of the weight function

Generating function $A_{sh}(x;r)$ of the shifted Narayana polynomials can be expressed in terms of series reversion. For more details, see [1].

Lemma 3.1. *Denote by $f(u)$ the following function*

$$v = f(u) = \frac{u}{1+(r+1)u+ru^2}. \tag{11}$$

Series reversion of $f(u)$ satisfies

$$u = f^{-1}(v) = \sum_{n=0}^{+\infty} a_{sh}(n;r)v^{n+1}. \tag{12}$$

Denote by $A_{sh}(x;r)$ the generating function of the sequence $a_{sh}(n;r)$, i.e.

$$A_{sh}(x;r) = \sum_{n=0}^{+\infty} a_{sh}(n;r)x^n \ .$$

From Lemma 3.1, we have that $A_{sh}(x;r)$ satisfies the following equation:

$$\frac{xA_{sh}(x;r)}{1+(r+1)xA_{sh}(x;r)+rx^2A^2(x;r)} = x \ .$$

This equation has two solutions for $A_{sh}(x;r)$, but only one satisfies the condition $A_{sh}(0;r) = a_{sh}(0;r) = 1$. Hence, we obtain

$$A_{sh}(x;r) = \frac{1-(r+1)x-\sqrt{(1-(r+1)x)^2-4rx^2}}{2rx^2} \ .$$

Our next goal is to find weight function $w_{sh}(x)$ whose moments are $a_{sh}(n;r)$, i.e.

$$a_{sh}(n;r) = \int_{\mathbb{R}} x^n w_{sh}(x)dx \ . \tag{13}$$

Lemma 3.2. *The weight function whose n-th moment is $a_{sh}(n;r)$ is*

$$w_{sh}(x) = \begin{cases} \frac{\sqrt{4r-(x-r-1)^2}}{2\pi r} & , \quad x \in \left((\sqrt{r}-1)^2, (\sqrt{r}+1)^2\right), \\ 0 & , \quad \text{otherwise} \ . \end{cases} \tag{14}$$

Proof. The proof is based on Stieltjes-Perron inversion formula (Theorem 2.1). At first, let us denote by $F(z,r)$ the following function:

$$F(z;r) = z^{-1}A_{sh}\left(z^{-1};r\right) = -\frac{(r+1)-z+\sqrt{(z-r-1)^2-4r}}{2rz} \ .$$

and with $\psi(x)$ the distribution function given by:

$$\psi(x)-\psi(0) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_0^t \Im F(x+iy;r)dx. \tag{15}$$

Consider the function $F(z;r)$ in the half-plane $\{z \in \mathbb{C} : \ \Im z > 0\}$.

The function

$$\rho(z;r) = \sqrt{(z-r-1)^2 - 4r}$$

has the branch points $(\sqrt{r}-1)^2$ and $(\sqrt{r}+1)^2$. We need to choose regular branch of square root such that it is positive when the expression under the square root is positive.

By explicit evaluation, we get that the integral of $F(z;r)$ is

$$\mathcal{F}(z;r) = \int F(z;r)dz = \frac{1}{4r}\Big(z^2 + (1+r-z)\rho(z;r) - 2z(r+1)\Big) + l_1(z;r) \ , \tag{16}$$

where

$$l_1(z;r) = \ln\big(-(r+1) + z + \rho(z;r)\big) \ .$$

Now equation (15) becomes

$$\psi(x) - \psi(0) = -\frac{1}{\pi}\lim_{y\to 0+}(\mathcal{F}(x+iy;r) - \mathcal{F}(iy;r)). \tag{17}$$

Let us now find $\lim_{y\to 0^+} \Im\mathcal{F}(x+iy;r)$. It is obvious that:

$$\lim_{y\to 0^+} \Im\rho(x+iy;r) = \begin{cases} \sqrt{4r-(x-r-1)^2} & , \quad x \in ((\sqrt{r}-1)^2, (\sqrt{r}+1)^2) \ , \\ 0 & , \quad \text{otherwise} \ . \end{cases}$$

The function $l_1(z;r)$ has one more branch point $z = r+1$. We take a branch of $l_1(z;r)$ such that imaginary part is 0 if the value under logarithm is real and positive. It holds:

$$\lim_{y\to 0^+} \Im\, l_1(x+iy;r) = \begin{cases} \pi + \arctan\frac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)} & , \quad x \in ((\sqrt{r}-1)^2, r+1) \ , \\ \arctan\frac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)} & , \quad x \in (r+1, (\sqrt{r}+1)^2) \ , \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Including two previous results in $\mathcal{F}(x+iy;r)$ and using (17), we get required expression for the weight function

$$w_{sh}(x) = \psi'(x) = \frac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, \qquad x \in \big((\sqrt{r}-1)^2, (\sqrt{r}+1)^2\big) \ .$$

Outside the segment $[(\sqrt{r}-1)^2, (\sqrt{r}+1)^2]$, we have $w_{sh}(x) = 0$. □

3.2 Evaluation of the coefficients of three-term recurrence relation

Denote by $\big(Q_n^{sh}(x)\big)_{n\in\mathbb{N}_0}$ the sequence of monic polynomials, orthogonal with respect to the weight function $w_{sh}(x)$.

Example 3.1. *The first few members of the sequence* $\big(Q_n^{sh}(x)\big)_{n\in\mathbb{N}_0}$ *are:*

$$Q_0^{sh}(x) = 1, \qquad \|Q_0^{sh}\|^2 = 1,$$
$$Q_1^{sh}(x) = x - (r+1), \qquad \|Q_1^{sh}\|^2 = r,$$
$$Q_2^{sh}(x) = x^2 - 2(r+1)x + r^2 + r + 1, \qquad \|Q_2^{sh}\|^2 = r^2.$$

In order to evaluate explicitly the coefficients $\left(\alpha_n^{sh}\right)_{n\in\mathbb{N}_0}$ and $\left(\beta_n^{sh}\right)_{n\in\mathbb{N}_0}$ of three-term recurrence relation

$$Q_{n+1}^{sh}(x) = (x - \alpha_n^{sh})Q_n^{sh}(x) - \beta_n^{sh}Q_{n-1}^{sh}(x), \tag{18}$$

we apply transformation formulas. These formulas connect the coefficients α_n^{sh} and β_n^{sh} of the original and transformed weight function.

Lemma 3.3. *Let $w(x)$ and $\bar{w}(x)$ be weight functions and denote by $(\pi_n(x))_{n\in\mathbb{N}_0}$ and $(\bar{\pi}_n(x))_{n\in\mathbb{N}_0}$ corresponding orthogonal polynomials. Also denote by $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ and $(\bar{\alpha}_n)_{n\in\mathbb{N}_0}$, $\left(\bar{\beta}_n\right)_{n\in\mathbb{N}_0}$ three-term relation coefficients corresponding to $w(x)$ and $\bar{w}(x)$ respectively. The following transformation formulas are valid:*

(1) *If $\bar{w}(x) = Cw(x)$ where $C > 0$ then holds $\bar{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\bar{\beta}_0 = C\beta_0$, $\bar{\beta}_n = \beta_n$ for $n \in \mathbb{N}$. Additionally holds $\bar{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.*

(2) *If $\bar{w}(x) = w(ax + b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\bar{\alpha}_n = \frac{\alpha_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\bar{\beta}_0 = \frac{\beta_0}{|a|}$ and $\bar{\beta}_n = \frac{\beta_n}{a^2}$ for $n \in \mathbb{N}$. Additionally holds $\bar{\pi}_n(x) = \frac{1}{a^n}\pi_n(ax + b)$.*

Proof. In both cases, we directly check the orthogonality of $\bar{\pi}_n(x)$ and obtain the coefficients $\bar{\alpha}_n$ and $\bar{\beta}_n$ by putting $\bar{\pi}_n(x)$ in the three-term recurrence relation for $\pi_n(x)$. □

Lemma 3.4. *The coefficients α_n^{sh} and β_n^{sh} $(n \in \mathbb{N}_0)$ in the three-term recurrence relation (18) are given by*

$$\beta_0^{sh} = 1, \quad \beta_n^{sh} = r \quad (n \in \mathbb{N}); \qquad \alpha_n^{sh} = r + 1 \qquad (n \in \mathbb{N}_0).$$

Proof. The monic Chebyshev polynomials of the second kind

$$Q_n^{(1)}(x) = S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1 - x^2}}$$

are orthogonal with respect to the weight $w^{(1)}(x) = \sqrt{1 - x^2}$. The corresponding coefficients in three-term relation are

$$\beta_0^{(1)} = \frac{\pi}{2}, \quad \beta_n^{(1)} = \frac{1}{4} \quad (n \geq 1), \qquad \alpha_n^{(1)} = 0 \quad (n \geq 0)\ .$$

Let us introduce new weight function $w^{(2)}(x) = \sqrt{4r - (x - r - 1)^2}$. It satisfies $w^{(2)}(x) = w^{(1)}(ax + b)$, where $a = 1/(2\sqrt{r})$ and $b = -(r + 1)/(2\sqrt{r})$. Hence we get (see Lemma 3.3):

$$\beta_0^{(2)} = \pi\sqrt{r}, \quad \beta_n^{(2)} = r \quad (n \in \mathbb{N}), \qquad \alpha_n^{(2)} = r + 1 \qquad (n \in \mathbb{N}_0)\ .$$

Finally, since $w_{sh}(x) = w^{(2)}(x)/(\pi\sqrt{r})$, we conclude that $\beta_n^{sh} = \beta_n^{(2)} = r$ for $n \geq 1$ and $\alpha_n^{sh} = \alpha_n^{(2)} = r + 1$. Coefficient β_0^{sh} can be obtained by direct evaluation of the integral $\beta_0^{sh} = \int_{\mathbb{R}} w_{sh}(x)dx = 1$. □

Corollary 3.5. *The squared norms of monic orthogonal polynomials $Q_n(x)$ have values*

$$\|Q_n^{sh}\|^2 = r^n \qquad (n \in \mathbb{N}_0)\ .$$

Proof. By using the statement of Lemma3.4 and formula (7), the conclusion follows. □

3.3 Proof of the main result

Denote by $(h_{sh}(n;r))_{n\in\mathbb{N}_0}$ the Hankel transform of the sequence $(a_{sh}(n;r))_{n\in\mathbb{N}_0}$. Now using Heilermann formula (7), we obtain:

$$h_{sh}(n;r) = a_{sh}(0;r)^{n+1}\prod_{i=1}^{n}\left(\beta_i^{sh}\right)^{n+1-i} = 1^{n+1}\prod_{i=1}^{n} r^{n+1-i} = r^{\binom{n+1}{2}} .$$

This completes the proof of the main result of this section.

Theorem 3.6. *The Hankel transform of the shifted Narayana polynomials* $(a_{sh}(n;r))_{n\in\mathbb{N}_0}$ *is given by*

$$h_{sh}(n;r) = r^{\binom{n+1}{2}} .$$

4 The Hankel transform of the Narayana and modified Narayana polynomials

Denote by $(h(n;r))_{n\in\mathbb{N}_0}$ and $(h_1(n;r))_{n\in\mathbb{N}_0}$, Hankel transforms of the Narayana and modified Narayana polynomials, i.e. $(a(n;r))_{n\in\mathbb{N}_0}$ and $(a_1(n;r))_{n\in\mathbb{N}_0}$.

From the relation

$$a_1(n;r) = a_{sh}(n-1;r) \qquad (n \geq 1), \tag{19}$$

we conclude that $(a_1(n;r))_{n\geq 1}$ can be expressed as the moment sequence, in the following form

$$a_1(n;r) = \int_{\mathbb{R}} x^n \tilde{w}(x)\, dx \quad (n \geq 1), \quad \text{where} \quad \tilde{w}(x) = \frac{w_{sh}(x)}{x}. \tag{20}$$

On the other side, there holds

$$\int_{\mathbb{R}} \frac{w_{sh}(x)}{x} dx = r^{-1} \neq a_1(0;r)$$

Hence, we introduce the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ defined by

$$\tilde{a}(n;r) := \begin{cases} r^{-1} , & n = 0 \\ a_1(n;r) , & n \geq 1 \end{cases} \tag{21}$$

which is the moment sequence for $\tilde{w}(x)$. Following lemma shows the connection between Hankel transforms of the sequences $(a_1(n;r))_{n\in\mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$.

Lemma 4.1. *For the sequences* $(a_k)_{n\in\mathbb{N}_0}$ *and* $(\tilde{a}_k)_{n\in\mathbb{N}_0}$ *, such that* $a_k = \tilde{a}_k$ *for all* $k \geq 1$*, their Hankel transforms* $(h_n)_{n\in\mathbb{N}_0}$ *and* $\left(\tilde{h}_n\right)_{n\in\mathbb{N}_0}$ *are related by*

$$h_n = \tilde{h}_n + (a_0 - \tilde{a}_0)\hat{h}_{n-1} \qquad (n \in \mathbb{N}_0) ,$$

where $\{\hat{h}_n\}_{n\in\mathbb{N}_0}$ *is the Hankel transform of the sequence* $\{\hat{a}_k\}$ *given by* $\hat{a}_k = a_{k+2}$ $(k \geq 0)$ *and* $\hat{h}_{-1} = 1$.

Proof. By expanding the determinant $h_n = \det[a_{i+j-2}]_{1\le i,j\le n}$ over the first row, we get

$$h_n = \sum_{k=0}^{n-1} a_k M_{1,k+1} \ , \tag{22}$$

where $M_{1,k}$ is the minor corresponding to the matrix element $(1,k)$. Also, we can write $\tilde{h}_n$ in the form

$$\tilde{h}_n = \sum_{k=0}^{n-1} \tilde{a}_k \tilde{M}_{1,k+1} \ . \tag{23}$$

Note that the minors $M_{1,k+1}$ and $\tilde{M}_{1,k+1}$ are equal for every $k \in \mathbb{N}_0$. Hence, we have

$$h_n - \tilde{h}_n = (a_0 - \tilde{a}_0) M_{1,1}.$$

But from the other side, $M_{1,1}$ is the $(n-1)$-th member of the Hankel transform of the sequence $\{a_2, a_3, \ldots\}$, i.e. it holds $M_{1,1} = \det[a_{i+j}]_{1\le i,j\le n-1}$ and we denote it by $\hat{h}_{n-1}$. □

Applying Lemma 4.1 on the sequences $(a_1(n;r))_{n\in\mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$, we obtain the following corollary.

Corollary 4.2. *If $h_1(n;r)$, $\tilde{h}(n;r)$ and $\hat{h}(n;r)$ are the Hankel transforms of the sequences $a_1(n;r)$, $\tilde{a}(n;r)$ and $\hat{a}(n;r) = a_1(n+2;r)$ respectively, then holds*

$$h_1(n;r) = \tilde{h}(n;r) + \left(1 - r^{-1}\right) \hat{h}(n-1;r) \ .$$

Now we have to evaluate the Hankel transform of the sequences $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$.

4.1 The Hankel transform of $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$

Note that $\hat{a}(n;r) = a_{sh}(n+1;r)$, i.e. $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$ is the shifted sequence of $(a_{sh}(n;r))_{n\in\mathbb{N}_0}$. Hence $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$ is the moment sequence of the weight function $\hat{w}(x) = x w_{sh}(x)$. The following lemma connects the Hankel transform of the original and shifted sequence.

Lemma 4.3. *Let $\omega(x)$ be weight function and $a_n = \int_{\mathbb{R}} x^n \omega(x)dx$ its n-th moment. Denote by $\bar{\omega}(x) = x\omega(x)$ and with $(Q_n(x))_{n\in\mathbb{N}_0}$ sequence of monic orthogonal polynomials corresponding to weight $\omega(x)$. Then the Hankel transforms $(h_n)_{n\in\mathbb{N}_0}$ and $\left(\bar{h}_n\right)_{n\in\mathbb{N}_0}$ of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(\bar{a}_n)_{n\in\mathbb{N}_0}$ are related by*

$$\bar{h}_n = h_n (-1)^{n+1} \lambda_{n+1} \qquad (n \in \mathbb{N}_0), \tag{24}$$

where $\lambda_n = Q_n(0)$ $(n \in \mathbb{N}_0)$.

Proof. The coefficients β_n and $\bar{\beta}_n$ satisfy (see [8] or [9]):

$$\bar{\beta}_n = \beta_n \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} \ .$$

Replacing in Heilermann formula (7), we obtain:

$$\bar{h}_{n+1} = \bar{h}_n \bar{a}_0 \prod_{i=1}^{n+1} \bar{\beta}_i = \bar{h}_n \bar{a}_0 \prod_{i=1}^{n+1} \beta_i \frac{\lambda_{i-1}\lambda_{i+1}}{\lambda_i^2} = h_{n+1} \frac{\bar{h}_n}{h_n} \left(\frac{\lambda_0 \bar{a}_0}{\lambda_1 a_0}\right) \frac{\lambda_{n+2}}{\lambda_{n+1}} \ . \tag{25}$$

By applying previous equation n times, we have:

$$\frac{\bar{h}_{n+1}}{h_{n+1}} = \frac{\bar{h}_0}{h_0}\left(\frac{\bar{a}_0}{a_0}\right)^{n+1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n+1}\frac{\lambda_{n+2}}{\lambda_1} . \tag{26}$$

Now, since $\bar{h}_0 = \bar{a}_0 = a_1$, $h_0 = a_0$ and $\lambda_0 = 1$, we have

$$\bar{h}_{n+1} = h_{n+1}\left(\frac{a_1}{\lambda_1 a_0}\right)^{n+2}\lambda_{n+2}.$$

Also note that $\lambda_1 = Q_1(0) = -a_1/a_0$, so $\bar{h}_{n+1} = (-1)^{n+2}h_{n+1}\lambda_{n+1}$. After decreasing index by 1 we finally get desired relation (24). Note that for $n = 0$ relation (24) trivially holds. □

By applying Lemma 4.3 on the sequences $(a_{sh}(n;r))_{n\in\mathbb{N}_0}$ and $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$, we evaluate Hankel transform of $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$.

Corollary 4.4. *The Hankel transform of the sequence $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$ is given by*

$$\hat{h}(n;r) = r^{\binom{n+1}{2}}\ \frac{r^{n+2}-1}{r-1} \qquad (n \in \mathbb{N}_0) .$$

Proof. Recall that, from Lemma 3.4 we have

$$\beta_0^{sh} = 1, \quad \beta_n^{sh} = r \quad (n \in \mathbb{N}); \qquad \alpha_n^{sh} = r+1 \quad (n \in \mathbb{N}_0).$$

Denote by $\lambda_n = Q_n^{sh}(0)$. Then holds

$$\lambda_{n+1} + (r+1)\lambda_n + r\lambda_{n-1} = 0 \quad (n \in \mathbb{N}_0), \qquad \lambda_0 = 1, \quad \lambda_1 = -(r+1) .$$

Previous equation has a unique solution $\lambda_n = (-1)^n(r^{n+1}-1)/(r-1)$. Now we obtain required formula for $\hat{h}(n;r)$ using $h_{sh}(n;r) = r^{\binom{n+1}{2}}$ (Theorem 3.6) and Lemma 4.3. □

4.2 The Hankel transform of $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$

Recall that $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ is defined as the moment sequence of $\tilde{w}(x) = w_{sh}(x)/x$. The following lemma is proved in [9] and establishes required transformation formulas in this case.

Lemma 4.5. [9] *Consider the same notation as in the Lemma 3.3. Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ is defined by*

$$r_{-1} = -\int_{\mathbb{R}} \bar{w}(x)\ dx, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \qquad (n \in \mathbb{N}_0). \tag{27}$$

If $\bar{w}(x) = \frac{w(x)}{x-c}$ where $c < \inf\operatorname{supp}(w)$ there holds

$$\begin{aligned} &\bar{\alpha}_0 = \alpha_0 + r_0, &&\bar{\alpha}_n = \alpha_n + r_n - r_{n-1}, \\ &\bar{\beta}_0 = -r_{-1}, &&\bar{\beta}_n = \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \qquad (n \in \mathbb{N}). \end{aligned} \tag{28}$$

Now we are ready to evaluate the Hankel transform of the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$.

Lemma 4.6. *The coefficients $\tilde{\beta}_n$ are given by*

$$\tilde{\beta}_n = \begin{cases} \beta_{n-1}, & n > 1\ ; \\ r^{-1}, & n = 0\ . \end{cases}$$

The Hankel transform $\left(\tilde{h}(n;r)\right)_{n\in\mathbb{N}_0}$ of the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ is given by

$$\tilde{h}(n;r) = r^{\binom{n}{2}-1}\ .$$

Proof. We apply Lemma 4.5. The coefficients $\tilde{\beta}_n$ are given by $\tilde{\beta}_n = \beta^{sh}_{n-1}\frac{r_{n-1}}{r_{n-2}}$, where sequence $(r_n)_{n\in\mathbb{N}_0}$ is determined by following recurrence relation:

$$r_n = -\alpha^{sh}_n - \frac{\beta^{sh}_n}{r_{n-1}}, \quad r_{-1} = -a_0\ . \tag{29}$$

After exchanging α^{sh}_n and β^{sh}_n from Lemma 3.4, we have

$$r_n + r + 1 + \frac{r}{r_{n-1}} = 0 \quad (n \geq 1), \qquad r_0 = -1, \quad r_{-1} = -r^{-1}\ . \tag{30}$$

It can be proven by mathematical induction that $r_n = -1$ for all $n \geq 1$. Hence

$$\tilde{\beta}_n = \beta^{sh}_{n-1}\frac{r_{n-1}}{r_{n-2}} = r \quad (n > 1), \quad \tilde{\beta}_1 = \beta^{sh}_0\frac{r_0}{r_{-1}} = r, \quad \tilde{\beta}_0 = \int_{\mathbb{R}} \frac{w_{sh}(x)}{x}dx = r^{-1}\ .$$

Now, the Hankel transform is:

$$\tilde{h}(n;r) = \tilde{a}_0^{n+1}\tilde{\beta}_1^n\tilde{\beta}_2^{n-1}\dots\tilde{\beta}_{n-1}^2\tilde{\beta}_n = \frac{1}{r^{n+1}}r^{\binom{n+1}{2}} = r^{\binom{n}{2}-1}\ .\square$$

4.3 Proof of the main result

From the previous facts, we can formulate and prove the main statement od this section.

Theorem 4.7. *The Hankel transform $(h_1(n;r))_{n\in\mathbb{N}_0}$ of the sequence of modified Narayana polynomials $(a_1(n;r))_{n\in\mathbb{N}_0}$ given by*

$$h_1(n;r) = r^{\binom{n+1}{2}}. \tag{31}$$

Proof. From Corollary 4.2 we have:

$$h_1(n;r) = \tilde{h}(n;r) + \left(1 - r^{-1}\right)\hat{h}(n-1;r) = r^{\binom{n+1}{2}-n} + \left(1 - r^{-1}\right)r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1}$$

$$= r^{\binom{n}{2}-1} + \frac{r-1}{r}r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1} = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}.\square$$

The evaluation of the Hankel transform of the sequence of Narayana polynomials $(a(n;r))_{n\in\mathbb{N}_0}$, now goes straightforward.

Theorem 4.8. *The Hankel transform $(h(n;r))_{n\in\mathbb{N}_0}$ of the sequence of Narayana polynomials $(a(n;r))_{n\in\mathbb{N}_0}$ given by*

$$h(n;r) = r^{\binom{n+1}{2}}. \tag{32}$$

Proof. Notice that $a(n;r) = r\tilde{a}(n;r)$ for any $n \geq 0$. Therefore it can be easily concluded that the Hankel transform of Narayana polynomials $(a(n;r))_{n\in\mathbb{N}_0}$ is given by

$$h(n;r) = r^{n+1}\tilde{h}(n;r) = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}.\square \tag{33}$$

4.4 Special cases

Now let us recall few special cases mentioned in the first section:

Example 4.1. In the case $r = 1$ we confirm known result that Hankel transform of the Catalan sequence $\mathcal{C}_n = a_1(n;1) = a(n;1)$. After exchanging $r = 1$ in (32) we can easily establish well-known result ([19] for example):

$$h_1(n;1) = h(n;1) = 1 \qquad (n \in \mathbb{N}_0).$$

Example 4.2. Recall also that in the case $r = 2$, we have that $(a_1(n;2))_{n\in\mathbb{N}_0}$ is the sequence of little Schröder numbers $(1, 1, 3, 11, 45, 197, 903, 4279, \ldots)$. Again, by exchanging $r = 2$ in (32) we have the following result:

$$h_1(n;2) = 2^{\binom{n+1}{2}} \qquad (n \in \mathbb{N}_0).$$

Example 4.3. Numbers $(a(n;2))_{n\in\mathbb{N}_0}$ are large Schröder numbers: $(1, 2, 6, 22, 90, 394, \ldots)$. By exchanging $r = 2$ in (33) we also obtain the following well-known result [3]:

$$h(n;2) = 2^{\binom{n+1}{2}} \qquad (n \in \mathbb{N}_0).$$

5 The Hankel transform of the sums of two consecutive shifted Narayana polynomials

In this section we find the closed-form expression for the Hankel transform of sequence $s(n;r) = a_{sh}(n;r) + a_{sh}(n+1;r)$. This result is similar to the one derived in [18]. It is clear that the elements of the sequence $s(n;r)$ are the moments corresponding to the weight function:

$$w_s(x) = (x+1)w_{sh}(x) = \begin{cases} \frac{(x+1)\sqrt{4r-(x-r-1)^2}}{2\pi r}, & x \in ((\sqrt{r}-1)^2, (\sqrt{r}+1)^2) \\ 0, & \text{otherwise} \end{cases}.$$

To find the Hankel transform of the sequence $(s(n;r))_{n\in\mathbb{N}_0}$, we need to construct orthogonal polynomials $(Q_n^s(x))_{n\in\mathbb{N}_0}$ corresponding to the weight $w_s(x)$ and to obtain the closed form of the coefficients of three-terms recurrence relation satisfied by these polynomials.

Lemma 5.1. *The coefficients α_n^s and β_n^s in three-term relation satisfied by $(Q_n^s(x))_{n\in\mathbb{N}_0}$ are:*

$$\beta_n^s = r\frac{(t_2^n - t_1^n)(t_2^{n+2} - t_1^{n+2})}{(t_2^{n+1} - t_1^{n+1})^2}, \qquad \alpha_n^s = -1 - \frac{t_2^{n+2} - t_1^{n+2}}{t_2^{n+1} - t_1^{n+1}} - r\frac{t_2^{n+1} - t_1^{n+1}}{t_2^{n+2} - t_1^{n+2}}, \tag{34}$$

where $t_{1,2} = \frac{-r-2\pm\sqrt{r^2+4}}{2}$ and $\beta_0^s = r + 2$.

Proof. By introducing a new weight function $w_s(x) = (x+1)w_{sh}(x)$ we can derive the coefficients α_n^s and β_n^s by using following relations:

$$\begin{aligned} \mu_n &= Q_n(-1), \\ \alpha_n^s &= -1 - \frac{\mu_{n+1}}{\mu_n} - \beta_{n+1}^{sh}\frac{\mu_n}{\mu_{n+1}}, \\ \beta_n^s &= \beta_n^{sh}\frac{\mu_{n-1}\mu_{n+1}}{\mu_n^2}. \end{aligned} \tag{35}$$

To obtain values $\mu_n = Q_n(-1)$, let us rewrite three-term recurrence relation for polynomials $Q_n(x)$ and take $x = -1$:

$$\mu_{n+1} + (r+2)\mu_n + r\mu_{n-1} = 0 \ . \tag{36}$$

Trivially we have: $\mu_{-1} = 0$ and $\mu_0 = 1$. Relation (36), together with the stated initial conditions is second order difference equation with constant coefficients. Hence, the solution of (36) is given by

$$\mu_n = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1} \ . \tag{37}$$

By replacing (37) in (35), we finally obtain expressions (34). □

Now we are ready to apply Heilermann formula and to prove the main result of this section:

$$\begin{aligned} h_s(n;r) &= h_s(n-1;r) \cdot s(0;r)\beta_n^s \dots \beta_1^s \\ &= h_s(n-1;r) \cdot s(0;r)\beta_n^{sh} \dots \beta_1^{sh} \cdot \frac{\mu_{n+1}\mu_{n-1}}{\mu_n^2} \cdot \frac{\mu_n\mu_{n-2}}{\mu_{n-1}^2} \cdots \frac{\mu_2\mu_0}{\mu_1^2} \\ &= h_s(n-1;r) \cdot \frac{h_{sh}(n;r)}{h_{sh}(n-1;r)} \cdot \frac{\mu_{n+1}}{\mu_n} \cdot \frac{s(0;r)\mu_0}{a_{sh}(0;r)\mu_1} \end{aligned}$$

By successive application of the previous equation we obtain

$$h_s(n;r) = h_s(0;r) \cdot \frac{h_{sh}(n;r)}{h_{sh}(0;r)} \cdot \frac{\mu_{n+1}}{\mu_1} \cdot \left(\frac{s(0;r)\mu_0}{a_{sh}(0;r)\mu_1}\right)^n \ ,$$

wherefrom

$$h_s(n;r) = h_{sh}(n;r) \cdot \mu_{n+1} \cdot \left(\frac{s(0;r)\mu_0}{a_{sh}(0;r)\mu_1}\right)^{n+1} \ . \tag{38}$$

Now by replacing:

$$h_s(0;r) = s(0;r) = a_{sh}(0;r) + a_{sh}(1;r) = r+2, \qquad \mu_1 = t_1 + t_2 = -(r+2), \qquad \mu_0 = 1,$$

and using formulas (32) and (37), we obtain:

$$h_s(n;r) = (-1)^{n+1} r^{\binom{n+1}{2}} \frac{t_2^{n+2} - t_1^{n+2}}{t_2 - t_1} \ .$$

Including the values for $t_{1,2}$ from Lemma 5.1, we complete the proof of the theorem:

Theorem 5.2. *The Hankel transform of the sequence $(s(n;r))_{n\in\mathbb{N}_0}$ is*

$$h_s(n;r) = \frac{r^{\binom{n+1}{2}}}{2^{n+2}\sqrt{r^2+4}} \left[\left(r+2+\sqrt{r^2+4}\right)^{n+2} - \left(r+2-\sqrt{r^2+4}\right)^{n+2}\right] \ .$$

The following lemma generalizes above approach and can be proven analogously as Theorem 5.2.

Lemma 5.3. *Let $\omega(x)$ be weight function and $a_n = \int_{\mathbb{R}} x^n \omega(x)dx$ its moments. Denote by $(Q_n(x))_{n\in\mathbb{N}_0}$ the sequence of orthogonal polynomials with respect to the weight $\omega(x)$. Also denote by $(h_n)_{n\in\mathbb{N}_0}$ and $(h_n^s)_{n\in\mathbb{N}_0}$, Hankel transforms of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(a_n + Ca_{n+1})_{n\in\mathbb{N}_0}$. The following relation is true:*

$$h_n^s = h_n \mu_{n+1} \left(\frac{Ca_0 + a_1}{a_0\mu_1}\right)^{n+1} , \tag{39}$$

where $\mu_n = Q_n(-C)$.

Especially, for shifted Narayana polynomials $a_{sh}(n;r)$, we find the Hankel transform of $s(n;r,C) = a_{sh}(n;r) + Ca_{sh}(n+1;r)$ using the formula (39). Sequence $(\mu_n)_{n\in\mathbb{N}_0}$ satisfies the following linear difference equation:

$$\mu_{n+1} + (r+1+C)\mu_n + r\mu_{n-1} = 0,$$

with the initial conditions $\mu_0 = 1$ and $\mu_{-1} = 0$. Solution of last equation has also the form (37), where coefficients t_1 and t_2 are given by

$$t_{1,2} = \frac{-r-1-C \pm \sqrt{(r+1+C)^2 - 4r}}{2}.$$

Also, it holds $\mu_1 = -a_{sh}(0;r) - C_s a_{sh}(1;r)$ implying:

$$h_s(n;r,C) = \frac{r^{\binom{n+1}{2}}}{2^{n+2}\sqrt{(r+1+C)^2-4r}}$$
$$\times \left(\left(r+1+C+\sqrt{(r+1+C)^2-4r}\right)^{n+2} - \left(r+1+C-\sqrt{(r+1+C)^2-4r}\right)^{n+2}\right).$$

Acknowledgement

This research was supported by the Science Foundation of Republic Serbia, Project No. 144023 and Project No. 144011.

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