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ELSEVIER Applied Mathematics and Computation 168 (2005) 1432–1448
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

On q-Newton–Kantorovich method for solving systems of equations

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Abstract

Starting from q-Taylor formula for the functions of several variables and mean value theorems in q-calculus which we prove by ourselves, we develop a new methods for solving the systems of equations. We will prove its convergence and we will give an estimation of the error.

Keywords: Iterative methods; q-derivative; q-Taylor formula

1. Introduction

At the last quarter of XX century, q-calculus appears as a connection between mathematics and physics (see [3–7]). It has a lot of applications in different mathematical areas, such as: number theory, combinatorics, orthogonal

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2004.10.035

polynomials, basic hyper geometric functions and other sciences: quantum theory, mechanics and theory of relativity.

Let q be a positive real number, $q \neq 1$. A q-complex number $[a]_q$ is

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}.$$

The q-factorial of a positive integer $[n]_q$ and q-binomial coefficient we define by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Also, q-Pochammer symbol is

$$(z-a)^{(0)} = 1, \quad (z-a)^{(k)} = \prod_{i=0}^{k-1}(z-aq^i) \quad (k \in \mathbb{N}). \tag{1.1}$$

2. On q-partial derivatives and differential

Let $f(\vec{x})$, where $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a multivariable real continuous function. We introduce an operator $\varepsilon_{q,i}$ which multiplies a coordinate of the argument by

$$(\varepsilon_{q,i}f)(\vec{x}) = f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n).$$

Furthermore,

$$(\varepsilon_q f)(\vec{x}) := (\varepsilon_{q,1}, \ldots, \varepsilon_{q,n} f)(\vec{x}) = f(q\vec{x}).$$

We define q-partial derivative of a function $f(\vec{x})$ to a variable x_i by

$$D_{q,x_i} f(\vec{x}) := \frac{f(\vec{x}) - (\varepsilon_{q,i}f)(\vec{x})}{(1-q)x_i} \quad (x_i \neq 0),$$

$$D_{q,x_i} f(\vec{x})|_{x_i=0} := \lim_{x_i \to 0} D_{q,x_i} f(\vec{x}).$$

At the similar way, high q-partial derivatives are

$$D_q^0 f(\vec{x}) = f(\vec{x}),$$

$$D^m_{q,x_1^{k_1},\ldots,x_i^{k_i},\ldots,x_n^{k_n}} f(\vec{x}) = D_{q,x_i}\left(D^{m-1}_{q,x_1^{k_1},\ldots,x_i^{k_i-1},\ldots,x_n^{k_n}} f(\vec{x})\right),$$
$$(k_1 + \cdots + k_n = m, \ m = 1, 2, \ldots).$$

Obviously,

$$D^{m+n}_{q,x_i^m x_j^n} f(\vec{x}) = D^{m+n}_{q,x_j^n x_i^m} f(\vec{x}) \quad (i, j = 1, 2 \ldots, n, \ m, n = 0, 1, \ldots).$$

Also, for an arbitrary $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, we can introduce q-differential

$$d_q f(\vec{x}, \vec{a}) := (x_1 - a_1) D_{q,x_1} f(\vec{a}) + (x_2 - a_2) D_{q,x_2} f(\vec{a}) + \cdots + (x_n - a_n) D_{q,x_n} f(\vec{a}),$$

and high q-differentials:

$$d_q^k f(\vec{x}, \vec{a}) := \left((x_1 - a_1) D_{q,x_1} + (x_2 - a_2) D_{q,x_2} + \cdots + (x_n - a_n) D_{q,x_n}\right)^{(k)} f(\vec{a}) = \sum_{\substack{i_1 + \cdots + i_n = k \\ i_j \in \mathbb{N}_0}} \frac{[k]_q!}{[i_1]_q! [i_2]_q! \cdots [i_n]_q!} D^k_{q,x_i^{i_1} \ldots x_n^{i_n}} f(\vec{a}) \prod_{j=1}^{n} (x_j - a_j)^{(i_j)}.$$

Notice, that a continuous function $f(\vec{x})$ in a neighborhood, which does not include any point with a zero coordinate, has also continuous q-partial derivatives.

3. About q-Taylor formula for a multivariable function

Now, we will discuss a new expansion of a function whose domain is a subset of $\mathbb{R}^n$. First of all, we need the next lemma.

Lemma 3.1. *It is valid*

$$D_{q,x}(x - \alpha)^{(n)} = [n]_q (x - \alpha)^{(n-1)} \quad (x, \alpha \in \mathbb{R},\ n \in \mathbb{N}).$$

For the proof see, for example, J. Cigler [2].

Theorem 3.2. *Suppose that all q-differentials of $f(x, y)$ exist in some neighborhood of (a, b). Then*

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{D^n_{q,x^i y^{n-i}} f(a, b)}{[i]_q! [n-i]_q!} (x - a)^{(i)} (y - b)^{(n-i)}.$$

Proof. Suppose that the function can be written in the next form

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} c_{n,i} (x - a)^{(i)} (y - b)^{(n-i)}.$$

Application of q-partial derivative operators $D_{q,x}$ and $D_{q,y}$ gives us

$$D^{k+m}_{q,x^k y^m} f(x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} c_{n,i} D^{k+m}_{q,x^k y^m} (x - a)^{(i)} (y - b)^{(n-i)}.$$

According to previous lemma, we conclude

$$D^{k+m}_{q,x^k y^m}(x-a)^{(i)}(y-b)^{(n-i)} = 0 \quad (k > i \vee m > n-i).$$

In other cases, we have

$$\begin{aligned}D^{k+m}_{q,x^k y^m}&(x-a)^{(i)}(y-b)^{(n-i)}\\ &= [i]_q \cdots [i-k+1]_q (x-a)^{(i-k)}\ [n-i]_q \cdots [n-i-m+1]_q (y-b)^{(n-i-m)}.\end{aligned}$$

Supposed expansion is valid in some neighborhood of (a,b). Putting $x = a$ and $y = b$, all members of the sum vanish, except for $i = k$ and $n - i = m$. Hence,

$$D^{k+m}_{q,x^k y^m} f(a,b) = c_{k+m,k}\ [k]_q!\ [m]_q! \qquad \square$$

In the same manner, we can prove the analogous theorem for the general case.

Theorem 3.3. *Suppose that there exist all q-differentials of $f(\vec{x})$ in some neighborhood of $\vec{a}$. Then*

$$f(\vec{x}) = \sum_{k=0}^{\infty} \frac{d^k_q f(\vec{x},\vec{a})}{[k]_q!}.$$

Unfortunately, it is very difficult to present the remainder term in q-Taylor formula for the functions of several variables in an operative form. However, for our further considerations it will be sufficient to formulate and prove the next theorem.

Theorem 3.4 (q-Lagrange). *Let $f(\vec{x})$ be a continuous function which has q-partial derivatives with respect to all variables x_j $(j = 1,\ldots,n)$ in some neighborhood B of $\vec{a}$. Let $\vec{x} \in B$ and $G = \{\vec{y} \in B : \|\vec{y}-\vec{a}\| \leqslant \|\vec{x}-\vec{a}\|\}$. Then there exists $\hat{q} \in (0,1)$ such*

$$\left(\forall q \in (\hat{q},1) \cup (1,\hat{q}^{-1})\right)\left(\exists \vec{\xi}^{(1)},\ldots,\vec{\xi}^{(n)} \in G\right):$$

$$f(\vec{x}) - f(\vec{a}) = \sum_{i=1}^{n} D_{q,x_i} f(\vec{\xi}^{(i)})(x_i - a_i).$$

Proof. Let us write

$$\begin{aligned}f(\vec{x}) - f(\vec{a}) &= f(x_1,\ldots,x_2) - f(a_1,\ldots,a_2)\\ &= \sum_{i=1}^{n} (f(a_1,\ldots,a_{i-1},x_i,x_{i+1},\ldots,x_n)\\ &\quad - f(a_1,\ldots,a_{i-1},a_i,x_{i+1},\ldots,x_n)).\end{aligned}$$

According to the q-Lagrange theorem for the functions of one variable (see [8]), for every fixed $i \in \{1,\ldots,n\}$, there exist $q_i \in (0,1)$, such that for every $q \in (q_i,1) \cup (1,q_i^{-1})$ there exists a value ξ_i with the property $|\xi_i - a_i| < |x_i - a_i|$ for which it is valid:

$$f(a_1,\ldots,a_{i-1},x_i,x_{i+1},\ldots,x_n) - f(a_1,\ldots,a_{i-1},a_i,x_{i+1},\ldots,x_n)$$
$$= D_{q,x_i} f(a_1,\ldots,a_{i-1},\xi_i,x_{i+1},x_n)(x_i - a_i)$$

Taking $\vec{\xi}^{(i)} = (a_1,\ldots,a_{i-1},\xi_i,x_{i+1},x_n)$, we get

$$\|\vec{\xi}^{(i)} - \vec{a}\| \leqslant \|\vec{x} - \vec{a}\| \quad (i = 1,\ldots,n).$$

For $\hat{q} = \max\{q_1,\ldots,q_n\}$ the statement of theorem holds. □

4. On q-Newton–Kantorovich method

We consider a system of nonlinear equations

$$\vec{f}(\vec{x}) = \mathbf{0},$$

where $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}),\ldots,f_n(\vec{x}))$ with $\vec{x} = (x_1,x_2,\ldots,x_n)$, $n \in \mathbb{N}$. We will suppose that this system has an isolated real solution $\vec{\xi}$. Using q-Taylor series of the function $\vec{f}(\vec{x})$ around some value $\vec{x}^{(m)} \approx \vec{\xi}$, we have

$$f_i(\vec{\xi}) \approx f_i\left(\vec{x}^{(m)}\right) + \sum_{j=1}^{n} D_{q,x_j} f_i\left(\vec{x}^{(m)}\right)\left(\xi_j - x_j^{(m)}\right) \quad (i = 1,2,\ldots,n).$$

In the matrix form, we rewrite

$$\vec{f}(\vec{\xi}) \approx \vec{f}\left(\vec{x}^{(m)}\right) + W_q\left(\vec{x}^{(m)}\right)\left(\vec{\xi} - \vec{x}^{(m)}\right),$$

where

$$W_q(\vec{x}) = D_q \vec{f}(\vec{x}) = \left[D_{q,x_j} f_i(\vec{x})\right]_{n\times n}$$

is the Jacobi matrix of q-partial derivatives. If the matrix W_q is regular, there exists the inverse matrix W_q^{-1}, so that we can formulate q-Newton–Kantorovich method in the form (see, for example, [1])

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - W_q\left(\vec{x}^{(m)}\right)^{-1}\vec{f}\left(\vec{x}^{(m)}\right).$$

Theorem 4.1. *Let the function $\vec{f}(\vec{x})$ has q-partial derivatives to all variables x_j $(i,j = 1,\ldots,n)$ in a ball $K[\vec{x}^{(0)},R] = \{\vec{x} : \|\vec{x} - \vec{x}^{(0)}\| \leqslant R\}$. Suppose that the matrix $W_q(\vec{x})$ is regular in this ball and the conditions*

$$\|W_q(\vec{x}) - W_q(\vec{y})\| \leqslant L\|\vec{x} - \vec{y}\|, \tag{4.1}$$

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x} - \vec{y})\| \leqslant \frac{L}{2}\|\vec{x} - \vec{y}\|^2 \tag{4.2}$$

are satisfied for all $\vec{x}, \vec{y} \in K[\vec{x}^{(0)}, R]$ *and a constant* $L > 0$. *If there are fulfilled the inequalities*

$$\|W_q(\vec{x}^{(0)})^{-1}\| \leqslant b, \quad \|W_q(\vec{x}^{(0)})^{-1}\vec{f}(\vec{x}^{(0)})\| \leqslant a, \quad h = abL \leqslant 1/2$$

and

$$R > r = \frac{1 - \sqrt{1 - 2h}}{h} a,$$

then the sequence $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ *converges to the solution* $\vec{\xi} \in K[\vec{x}^{(0)}, r]$ *and it is valid*

$$\|\vec{\xi} - \vec{x}^{(m)}\| \leqslant \frac{a}{2^{m-1}}(2h)^{2^m - 1} \quad (m \in \mathbb{N}).$$

Proof. At the start, let us construct the sequences $\{h_k\}_{k\in N_0}$, $\{a_k\}_{k\in N_0}$, $\{b_k\}_{k\in N_0}$ and $\{r_k\}_{k\in N_0}$ by

$$a_{k+1} = \frac{h_k}{2(1 - h_k)} a_k, \quad b_{k+1} = \frac{b_k}{1 - h_k},$$
$$h_{k+1} = a_{k+1} b_{k+1} L, \quad r_{k+1} = \frac{1 - \sqrt{1 - 2h_{k+1}}}{h_{k+1}} a_{k+1}$$

with the starting values $h_0 = h$, $a_0 = a$, $b_0 = b$, $r_0 = r$. We will prove that the sequence $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ exists and for every $k \in \mathbb{N}_0$ there holds

$$\left\| W_q\left(\vec{x}^{(k)}\right)^{-1} \right\| \leqslant b_k, \tag{4.3}$$

$$\left\| W_q\left(\vec{x}^{(k)}\right)^{-1} \vec{f}\left(\vec{x}^{(k)}\right) \right\| \leqslant a_k, \tag{4.4}$$

$$h_k \leqslant 1/2, \tag{4.5}$$

$$K[\vec{x}^{(k)}, r_k] \subset K[\vec{x}^{(k-1)}, r_{k-1}]. \tag{4.6}$$

The statements holds for $k = 0$ with respect to the conditions of the theorem. Using the method of mathematical induction, suppose that the statements are valid for any $k \leqslant m$ and prove that they are valid for $k = m + 1$ too. By the definition of the sequence and the induction conjecture,

$$\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| = \left\| W_q\left(\vec{x}^{(m)}\right)^{-1} \vec{f}\left(\vec{x}^{(m)}\right) \right\| \leqslant a_m.$$

Since

$$r_m = \frac{1-\sqrt{1-2h_m}}{h_m} a_m \geqslant a_m,$$

it is valid

$$\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant r_m,$$

i.e. $\vec{x}^{(m+1)} \in K[\vec{x}^{(m)}, r_m] \subset K[\vec{x}^{(0)}, R]$. So, $W_q(\vec{x}^{(m+1)})$ exists and it is regular. Its inverse can be expressed in the form

$$W_q\left(\vec{x}^{(m+1)}\right)^{-1} = \left(I + W_q\left(\vec{x}^{(m)}\right)^{-1}\left(W_q\left(\vec{x}^{(m+1)}\right) - W_q\left(\vec{x}^{(m)}\right)\right)\right)^{-1} W_q\left(\vec{x}^{(m)}\right)^{-1}.$$

Because, from (4.1) it holds

$$\lambda = \left\|-W_q\left(\vec{x}^{(m)}\right)^{-1}\left(W_q\left(\vec{x}^{(m+1)}\right) - W_q\left(\vec{x}^{(m)}\right)\right)\right\| \leqslant b_m L \left\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\right\|$$
$$\leqslant b_m L a_m = h_m \leqslant 1/2,$$

using Neumann expansion we get

$$\left\|W_q\left(\vec{x}^{(m+1)}\right)^{-1}\right\| \leqslant \sum_{i=0}^{\infty}\left\|W_q\left(\vec{x}^{(m)}\right)^{-1}\left(W_q\left(\vec{x}^{(m+1)}\right) - W_q\left(\vec{x}^{(m)}\right)\right)\right\|^i$$
$$\left\|W_q\left(\vec{x}^{(m)}\right)^{-1}\right\| \leqslant \sum_{i=0}^{\infty} b_m \lambda^i = \frac{b_m}{1-\lambda} \leqslant \frac{b_m}{1-h_m} = b_{m+1},$$

what proves (4.3).

From the definition of the sequence it follows that

$$W_q(\vec{x}^{(m)})\left(\vec{x}^{(m+1)} - \vec{x}^{(m)}\right) = -\vec{f}(\vec{x}^{(m)}),$$

wherefrom, according to (4.2), it can be written

$$\left\|\vec{f}(\vec{x}^{(m+1)})\right\| = \left\|\vec{f}(\vec{x}^{(m+1)}) - \vec{f}(\vec{x}^{(m)}) - W_q(\vec{x}^{(m)})\left(\vec{x}^{(m+1)} - \vec{x}^{(m)}\right)\right\|$$
$$\leqslant \frac{L}{2}\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\|^2 \leqslant \frac{L}{2} a_m^2.$$

Hence,

$$\left\|W_q\left(\vec{x}^{(m+1)}\right)^{-1}\vec{f}\left(\vec{x}^{(m+1)}\right)\right\| \leqslant \left\|W_q(\vec{x}^{(m+1)})^{-1}\right\|\left\|\vec{f}\left(\vec{x}^{(m+1)}\right)\right\| \leqslant b_{m+1}\frac{L}{2}a_m^2$$
$$= \frac{b_m}{1-h_m}\frac{L}{2}a_m^2 = \frac{h_m}{2(1-h_m)}a_m = a_{m+1},$$

what proves (4.4).

The inequality (4.5) is obvious:

$$h_{m+1} = a_{m+1}b_{m+1}L = \frac{h_m}{2(1-h_m)}a_m\frac{b_m}{1-h_m}L = \frac{h_m^2}{2(1-h_m)^2} \leqslant \frac{1}{2}.$$

For the proof of (4.6) we consider $\vec{y} \in K[\vec{x}^{(m+1)}, r_{m+1}]$. Then,

$$\|\vec{y} - \vec{x}^{(m)}\| \leqslant \|\vec{y} - \vec{x}^{(m+1)}\| + \|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant r_{m+1} + a_m.$$

It can be shown that $r_{m+1} + a_m = r_m$, so $\vec{y} \in K[\vec{x}^{(m)}, r_m]$, i.e. the inclusion is valid.

Finally, by the construction of the mentioned sequences, it is clear that

$$a_{m+1} = \frac{h_m}{2(1-h_m)}a_m \leqslant \frac{1}{2}a_m,$$

so, $\lim_{m\to\infty} a_m = 0$. Further,

$$\lim_{m\to\infty} r_m = \lim_{m\to\infty} \frac{2a_m}{(1+\sqrt{1-2h_m})} = 0,$$

wherefrom we conclude that there exists a unique point $\vec{\xi} \in \cap_{m=0}^{\infty} K[\vec{x}^{(m)}, r_m]$ and

$$\vec{\xi} = \lim_{m\to\infty} \vec{x}^{(m)}.$$

Since

$$\lim_{m\to\infty} \left\|\vec{f}\left(\vec{x}^{(m+1)}\right)\right\| \leqslant \lim_{m\to\infty} \frac{L}{2}a_m^2 = 0,$$

we get $\|\vec{f}(\vec{\xi})\| = 0$, i.e. $\vec{f}(\vec{\xi}) = \mathbf{0}$.

For the last inequality, let us note that

$$\frac{h_m}{1-h_m} \leqslant 2h_m,$$

and then

$$a_{m+1} \leqslant a_m h_m, \quad h_{m+1} \leqslant 2h_m^2$$

for any $m \in \mathbb{N}$. Repeating this, we have

$$h_m \leqslant \frac{1}{2}(2h)^{2^m}$$

and

$$a_m \leqslant h_{m-1}h_{m-2}a_{m-2} \leqslant \cdots \leqslant h_{m-1}h_{m-2},\ldots,ha \leqslant \frac{a}{2^m}(2h)^{1+2+\cdots+2^{m-1}}$$
$$= \frac{a}{2^m}(2h)^{2^m-1}.$$

Because

$$\|\vec{\xi} - \vec{x}^{(m)}\| \leqslant r_m = \frac{2a_m}{(1+\sqrt{1-2h_m})} \leqslant 2a_m,$$

we have the estimation of error. □

We can formulate and prove one more theorem about the convergence of the method with some changed conditions.

Theorem 4.2. *Let the function $\vec{f}(\vec{x})$ be continuous and has q-partial derivatives to all variables x_j $(j = 1,\dots,n)$ in a ball $K[\vec{x}^{(0)}, R]$. Suppose that the matrix $W_q(\vec{x})$ is regular in this ball and the conditions*

$$\|W_q(\vec{x}) - W_q(\vec{y})\| \leqslant L\|\vec{x}-\vec{y}\|, \tag{4.7}$$

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x}-\vec{y})\| \leqslant L\|\vec{x}-\vec{y}\|^2, \tag{4.8}$$

$$\left\|W_q(\vec{x})^{-1}\right\| \leqslant b, \quad \left\|W_q\left(\vec{x}^{(0)}\right)^{-1}\vec{f}\left(\vec{x}^{(0)}\right)\right\| \leqslant a, \tag{4.9}$$

are satisfied for all $\vec{x},\vec{y} \in K[\vec{x}^{(0)}, R]$ and a constant $L > 0$, where $a, b > 0$ are the constants such that and $h = abL < 1$. Let

$$r = \frac{a}{1-h} < R.$$

Then the sequence $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ converges to the solution $\vec{\xi} \in K[\vec{x}^{(0)}, r]$ and it holds the estimation

$$\|\vec{\xi} - \vec{x}^{(m)}\| \leqslant \frac{ah^{2^m-1}}{1-h^{2^m}}.$$

Proof. At the start, we will prove that $\vec{x}^{(m)} \in K[\vec{x}^{(0)}, r]$ for all $m \in \mathbb{N}_0$. By the definition of the sequence and the condition (4.9), we have

$$\|\vec{x}^{(1)} - \vec{x}^{(0)}\| = \left\|W_q\left(\vec{x}^{(0)}\right)^{-1}\vec{f}\left(\vec{x}^{(0)}\right)\right\| \leqslant a < r.$$

Suppose that $\vec{x}^{(k)} \in K[\vec{x}^{(0)}, r]$ for $k \leqslant m$. Then

$$\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| = \left\|W_q\left(\vec{x}^{(m)}\right)^{-1}\vec{f}\left(\vec{x}^{(m)}\right)\right\| \leqslant b\left\|\vec{f}\left(\vec{x}^{(m)}\right)\right\|.$$

Since

$$W_q(\vec{x}^{(m-1)})\left(\vec{x}^{(m)} - \vec{x}^{(m-1)}\right) = -\vec{f}\left(\vec{x}^{(m-1)}\right),$$

then

$$\left\|\vec{f}\left(\vec{x}^{(m)}\right)\right\| = \left\|\vec{f}\left(\vec{x}^{(m)}\right) - \vec{f}\left(\vec{x}^{(m-1)}\right) - W_q(\vec{x}^{(m-1)})\left(\vec{x}^{(m)} - \vec{x}^{(m-1)}\right)\right\|$$

and, according to (4.8),

$$\left\|\vec{f}\left(\vec{x}^{(m)}\right)\right\| \leqslant L\|\vec{x}^{(m)} - \vec{x}^{(m-1)}\|^2.$$

Hence,

$$\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant bL\|\vec{x}^{(m)} - \vec{x}^{(m-1)}\|^2 = \frac{h}{a}\|\vec{x}^{(m)} - \vec{x}^{(m-1)}\|^2,$$

i.e.,

$$\frac{h}{a}\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant \left(\frac{h}{a}\|\vec{x}^{(m)} - \vec{x}^{(m-1)}\|\right)^2.$$

This, after repeating m times, gives

$$\frac{h}{a}\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant \left(\frac{h}{a}\|\vec{x}^{(1)} - \vec{x}^{(0)}\|\right)^{2^m} \leqslant h^{2^m},$$

where from

$$\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant ah^{2^m-1}.$$

Finally,

$$\begin{aligned}\|\vec{x}^{(m+1)} - \vec{x}^{(0)}\| &\leqslant \|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| + \|\vec{x}^{(m)} - \vec{x}^{(m-1)}\| + \cdots + \|\vec{x}^{(1)} - \vec{x}^{(0)}\| \\ &\leqslant a\left(h^{2^m-1} + h^{2^{m-1}-1} + \cdots + h + 1\right) \leqslant a\sum_{m=0}^{\infty} h^m = \frac{a}{1-h} = r,\end{aligned}$$

so, $\vec{x}^{(m+1)} \in K[\vec{x}^{(0)}, r]$.

Let us prove that $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ is a Cauchy sequence. Really, for any $k, m \in \mathbb{N}$ it is valid

$$\begin{aligned}\|\vec{x}^{(m+k)} - \vec{x}^{(m)}\| &\leqslant \|\vec{x}^{(m+k)} - \vec{x}^{(m+k-1)}\| + \|\vec{x}^{(m+k-1)} - \vec{x}^{(m+k-2)}\| \\ &\quad + \cdots \|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant ah^{2^m-1}\left(1 + h^{2^m} + \cdots + h^{2^m(2^{k-1}-1)}\right) \\ &\leqslant ah^{2^m-1}\sum_{k=0}^{\infty}\left(h^{2^m}\right)^k = \frac{ah^{2^m-1}}{1-h^{2^m}}.\end{aligned}$$

Since $0 < h < 1$, the last term can be arbitrary small for large enough $m \in \mathbb{N}$. Therefore, $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ is a Cauchy sequence and it converges to a point $\vec{\xi} \in K[\vec{x}^{(0)}, r]$. Let us prove that $\vec{f}(\vec{\xi}) = \mathbf{0}$. According to (4.7),

$$\left\|W_q\left(\vec{x}^{(m)}\right) - W_q\left(\vec{x}^{(0)}\right)\right\| \leqslant L\|\vec{x}^{(m)} - \vec{x}^{(0)}\| \leqslant Lr,$$

and then

$$\left\|W_q\left(\vec{x}^{(m)}\right)\right\| \leqslant Lr + \left\|W_q\left(\vec{x}^{(0)}\right)\right\| = C \quad (C = \text{const.}).$$

So, by the definition of the sequence,

$$\left\|\vec{f}(\vec{x}^{(m)})\right\| \leqslant \left\|-W_q\left(\vec{x}^{(m)}\right)\right\|\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\| \leqslant C\|\vec{x}^{(m+1)} - \vec{x}^{(m)}\|.$$

Since $f_i(\vec{x})$, $i = 1,\ldots,n$, are continuous, it holds

$$\lim_{m\to\infty}\left\|\vec{f}(\vec{x}^{(m)})\right\| = \left\|\vec{f}(\vec{\xi})\right\| = 0,$$

wherefrom we get

$$\vec{f}(\vec{\xi}) = \mathbf{0}.$$

Finally, if $k \to \infty$ in the inequality

$$\|\vec{x}^{(m+k)} - \vec{x}^{(m)}\| \leqslant \frac{ah^{2^m-1}}{1-h^{2^m}}$$

we get the estimation. □

The conditions of Theorems 4.1 and 4.2 are quite strong and it seems that it is difficult adjust them. But, the Lipschitz condition (4.1)–(4.7) for $W_q(\vec{x})$ and inequality (4.8) can be presented over q-partial derivatives of the second order. Without loss of generality, let us choice following norms for vectors $\vec{x} = [x_1, \ldots, x_n]^{\mathrm{T}}$ and matrices $A = [a_{ij}]_{i,j=1}^n$:

$$\|\vec{x}\| = \max_{1\leqslant i\leqslant n}|x_i|, \quad \|A\| = \max_{1\leqslant i\leqslant n}\sum_{j=1}^n |a_{ij}|,$$

Lemma 4.3. *Let the function $\vec{f}(\vec{x})$ has q-partial derivatives of the second order in a ball K. If*

$$\sum_{k=1}^n\left|D^2_{q,x_jx_k}f_i(\vec{x})\right| \leqslant N, \quad (i,j = 1,\ldots,n,\ q\neq 1)$$

for a constant $N > 0$ and all $\vec{x}\in K$ then exists $\hat{q}\in(0,1)$ such that for every value $q\in(\hat{q},1)\cup(1,\hat{q}^{-1})$ and for all $\vec{x},\vec{y}\in K$ the conditions

$$\|W_q(\vec{x}) - W_q(\vec{y})\| \leqslant L\|\vec{x}-\vec{y}\|,$$

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x}-\vec{y})\| \leqslant L\|\vec{x}-\vec{y}\|^2$$

with $L = nN$ are satisfied.

Proof. By the accepted norm and the definition of $W_q(\vec{x})$, we have

$$\|W_q(\vec{x}) - W_q(\vec{y})\| = \max_{1\leqslant i\leqslant n}\sum_{j=1}^n\left|D_{q,x_j}f_i(\vec{x}) - D_{q,x_j}f_i(\vec{y})\right|.$$

Applying q-Lagrange mean value theorem for the functions of several variables to the functions $D_{q,x_j} f_i(\vec{x})$, there exist $q_{ij} \in (0,1)$ such that

$$\Big(\forall q \in (q_{ij},1) \cup (1, q_{ij}^{-1})\Big)\Big(\exists \vec{z}^{(i,1)}, \ldots, \vec{z}^{(i,n)} \in \{\vec{z} \,|\, \|\vec{z}-\vec{y}\| \leqslant \|\vec{x}-\vec{y}\|\}\Big)$$

$$: D_{q,x_j} f_i(\vec{x}) - D_{q,x_j} f_i(\vec{y}) = \sum_{k=1}^{n} D^2_{q,x_i x_k} f_i(\vec{z}^{(i,k)})(x_k - y_k).$$

If (4.3) is valid, then

$$\|W_q(\vec{x}) - W_q(\vec{y})\| \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} D^2_{q,x_j x_k} f_i(\vec{z}^{(i,k)})(x_k - y_k) \right|$$

$$\leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} D^2_{q,x_j x_k} f_i(\vec{z}^{(i,k)}) \right| \|\vec{x}-\vec{y}\| \leqslant nN\|\vec{x}-\vec{y}\|$$

for all $q \in (q_*, 1) \cup (1, q_*^{-1})$, where $q^* = \max_{1\leqslant i,j\leqslant n}\{q_{ij}\}$.

Further, applying q-Lagrange mean value theorem for the functions of several variables to the functions $f_i(\vec{x})$, there exist $\bar{q}_i \in (0,1)$ such that

$$\big(\forall q \in (\bar{q}_i,1) \cup (1, \bar{q}_i^{-1})\big)\Big(\exists \vec{u}^{(i,1)}, \ldots, \vec{u}^{(i,n)} \in \{\vec{u} \,|\, \|\vec{u}-\vec{y}\| \leqslant \|\vec{x}-\vec{y}\|\}\Big)$$

$$: f_i(\vec{x}) - f_i(\vec{y}) = \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{u}^{(i,j)})(x_j - y_j),$$

wherefrom it can be written

$$f_i(\vec{x}) - f_i(\vec{y}) - \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{y})(x_j - y_j)$$

$$= \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{u}^{(i,j)})(x_j - y_j) - \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{y})(x_j - y_j)$$

$$= \sum_{j=1}^{n} \Big(D_{q,x_j} f_i(\vec{u}^{(i,j)}) - D_{q,x_j} f_i(\vec{y}) \Big)(x_j - y_j).$$

Then

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x}-\vec{y})\| = \max_{1\leqslant i\leqslant n} \left| f_i(\vec{x}) - f_i(\vec{y}) - \sum_{j=1}^{n} D_{q,x_j} f_i(\vec{y})(x_j - y_j) \right|$$

$$\leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \left| D_{q,x_j} f_i(\vec{u}^{(i,j)}) - D_{q,x_j} f_i(\vec{y}) \right| |x_j - y_j|.$$

Applying q-Lagrange mean value theorem for the functions of several variables to the functions $D_{q;x_j} f_i(\vec{x})$, there exist $\tilde{q}_{ij} \in (0,1)$ such that

$$\left(\forall q \in (\tilde{q}_{ij}, 1) \cup (1, \tilde{q}_{ij}^{-1})\right)\left(\exists \vec{v}^{(i,j,1)}, \ldots, \vec{v}^{(i,j,n)} \in \{\vec{v} \,|\, \|\vec{v} - \vec{y}\| \leqslant \|\vec{x} - \vec{y}\|\}\right)$$
$$: D_{q,x_j} f_i(\vec{u}^{(i,j)}) - D_{q,x_j} f_i(\vec{y}) = \sum_{k=1}^{n} D^2_{q,x_j x_k} f_i(\vec{v}^{(i,j,k)})(u_k^{(i,j)} - y_k).$$

Hence

$$\begin{aligned}
&\|\vec{f}(\vec{x}) - \vec{f}(\vec{y}) - W_q(\vec{y})(\vec{x} - \vec{y})\| \\
&\quad \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} D^2_{q,x_j x_k} f_i(\vec{v}^{(i,j,k)})(u_k^{(i,j)} - y_k) \right| \|\vec{x} - \vec{y}\| \\
&\quad \leqslant \|\vec{x} - \vec{y}\| \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left| D^2_{q,x_j x_k} f_i(\vec{v}^{(i,j,k)}) \right| |u_k^{(i,j)} - y_k| \\
&\quad \leqslant \|\vec{x} - \vec{y}\| \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \|\vec{u}^{(i,j)} - \vec{y}\| \sum_{k=1}^{n} \left| D^2_{q,x_j x_k} f_i(\vec{v}^{(i,j,k)}) \right| \leqslant nN\|\vec{x} - \vec{y}\|^2
\end{aligned}$$

for any $q \in (q_{**}, 1) \cup (1, q_{**}^{-1})$, where $q_{**} = \max_{1\leqslant i,j\leqslant n}\{\bar{q}_i, \tilde{q}_{ij}\}$.

Finally, $\hat{q} = \max\{q_*, q_{**}\}$. □

Now we are ready to give the theorem about convergence of the method with more effective checkable conditions.

Theorem 4.4. *Let the function $\vec{f}(\vec{x})$ be continuous and has all q-partial derivatives of the second order in a closed ball $K[\vec{x}^{(0)}, R]$. Suppose that exist the constants $a, b, L > 0$, where $h = abL < 1$, and $\bar{q} \in (0, 1)$, such that for all $q \in (\bar{q}, 1) \cup (1, \bar{q}^{-1})$ and $\vec{x} \in K[\vec{x}^{(0)}, R]$, it is valid*:

$$\sum_{k=1}^{n} \left| D^2_{q,x_j x_k} f_i(\vec{x}) \right| \leqslant \frac{L}{n}, \quad (i, j = 1, \ldots, n). \tag{4.10}$$

Also, let the matrix $W_q(\vec{x})$ be regular in $K[x^{(0)}, R]$ and has the properties

$$\left\| W_q\left(\vec{x}^{(0)}\right)^{-1} \vec{f}\left(\vec{x}^{(0)}\right) \right\| \leqslant a, \quad \left\| W_q(\vec{x})^{-1} \right\| \leqslant b.$$

If

$$r = \frac{a}{1 - h} < R,$$

then there exists $\hat{q} \in (0, 1)$ such that for any $q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})$ the sequence $\{\vec{x}^{(m)}\}_{m\in\mathbb{N}_0}$ converges to the solution $\vec{\xi} \in K[\vec{x}^{(0)}, r]$ and it holds the estimation

$$\|\vec{\xi} - \vec{x}^{(m)}\| \leqslant \frac{a h^{2^m - 1}}{1 - h^{2^m}}.$$

Proof. According to Lemma 4.3 and the condition (4.10), it exists $\tilde{q} \in (0,1)$ such that for all $q \in (\tilde{q},1) \cup (1,\tilde{q}_{-1})$ the conditions (4.7) and (4.8) are satisfied. Then we get the statement as the corollary of Theorem 4.2, with $\hat{q} = \max\{\tilde{q},\bar{q}\}$. □

5. Equations with infinite products

For computing the infinite product

$$f(t,q) = \prod_{n=1}^{\infty}(1-tq^n) \quad (t \in \mathbb{C})(|q|<1),$$

A.D. Sokal in [9] suggests a quadratically convergent algorithm based on the identity

$$f(t,q) = \sum_{m=0}^{\infty} \frac{(-t)^m q^{m(m+1)/2}}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

Here, we are interesting in finding the solutions of an equation

$$F(t) \equiv f(t,q) - a = 0,$$

for a fixed value q and known values $a \in \mathbb{C}$. Let us notice

$$D_{q,t}f(t,q) = \frac{-q}{(1-q)(1-tq)} f(t,q).$$

Applying our q-Newton method [10], we find iterative process

$$t_{k+1} = t_k + \frac{1-q}{q}(1-at_k)\left(1 - \frac{a}{f(t_k,q)}\right)$$

which leads us to the solution of the previous equation.

Now, there is no problem to use our considerations for solving the systems of the type

$$\vec{H}(f(\vec{x},q)) = 0.$$

We will demonstrate it in the seventh section.

6. Zeros of the functions defined via q-integrals

Let us consider the equation

$$F(x) \equiv \int_0^x h(t)\,\mathrm{d}_q t - a = x(1-q)\sum_{k=0}^{\infty} h(xq^k)q^k - a = 0,$$

where a and q are real numbers and $|q| < 1$.

Since $D_qF(x) \equiv h(x)$, we can apply q-Newton method

$$x_{n+1} = x_n - \frac{F(x_n)}{h(x_n)} \quad (n = 0, 1, \ldots),$$

with some initial value x_0 (for example, $x_0 = a$). Instead of q-integral we evaluate partial sum with a proper exactness. Now,

$$\lim_{n\to\infty} x_n = x.$$

7. Examples

Example 7.1. For the system of nonlinear equations

$$f_1(x,y) \equiv x^3 + y + y^2 - 1.44 = 0,$$
$$f_2(x,y) \equiv x^2 + x + y^3 - 2.41 = 0,$$

we have the next q-Jacobi matrix

$$W_q(x,y) = \begin{bmatrix} (1+q+q^2)x^2 & 1+(1+q)y \\ 1+(1+q)x & (1+q+q^2)y^2 \end{bmatrix}$$

and the second q-derivatives

$$D_{q,xx}f_1 = (1+q)(1+q+q^2)x, \quad D_{q,yy}f_2 = (1+q)(1+q+q^2)y,$$

$$D_{q,xy}f_1 = D_{q,yx}f_1 = D_{q,xy}f_2 = D_{q,yx}f_2 = 0, \quad D_{q,yy}f_1 = D_{q,xx}f_2 = 1+q.$$

Starting with initial values $x_0 = -1.2$, $y_0 = 1.3$, we are looking for the solution inside the ball whose center is (x_0, y_0) and radius is $R = 0.1$. Applying method for $q = 0.9$, we have $b = 0.360752$, $a = 0.0407325$ and $N = 8.4$, $L = nN = 16.8$. Hence $h = abL = 0.246865 < 1/2$ and $r = 0.0475979 < R$, we conclude that q-Newton–Kantorovich method is converging. Really, we get the next iterations:

$$\vec{x}^{(k)} : \begin{bmatrix} -1.2 \\ 1.3 \end{bmatrix}, \begin{bmatrix} -1.15927 \\ 1.30549 \end{bmatrix}, \begin{bmatrix} -1.16194 \\ 1.30488 \end{bmatrix}, \begin{bmatrix} -1.16169 \\ 1.30495 \end{bmatrix}, \begin{bmatrix} -1.16171 \\ 1.30494 \end{bmatrix}.$$

Example 7.2. Let us consider the next system of nonlinear equations

$$x_1^2 + 7x_2 - x_3^4 = 2,$$
$$x_1^2 - 49x_2^2 + x_3^2 = 6,$$
$$x_1^2 + 7(x_2 - 1) - x_3^2 = -3.$$

If we use q-method, we yield the next q-Jacobi matrix

$$W_q(x_1,x_2,x_3) = \begin{bmatrix} (1+q)x_1 & 7 & -(1+q)(1+q^2)x_3^3 \\ (1+q)x_1 & -49(1+q)x_2 & (1+q)x_3 \\ (1+q)x_1 & 7 & -(1+q)x_3 \end{bmatrix}.$$

Using $q = 0.9$, we find the solutions ($x_1 = \sqrt{5}$, $x_2 = 1/7$, $x_3 = \sqrt{2}$), with accuracy on five decimal digits after $n = 7$ iterations.

$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.613 \\ 0.705 \\ 2.299 \end{bmatrix}, \begin{bmatrix} 2.199 \\ 0.353 \\ 1.747 \end{bmatrix}, \begin{bmatrix} 2.1794 \\ 0.1937 \\ 1.4633 \end{bmatrix}, \begin{bmatrix} 2.2331 \\ 0.1450 \\ 1.4078 \end{bmatrix} \to \begin{bmatrix} 2.23607 \\ 0.142871 \\ 1.41427 \end{bmatrix}.$$

The next example will show the advantages of q-Newton–Kantorovich method with respect to the classical one.

Example 7.3. Let us consider the next system of nonlinear equations

$$|x_1^2 - 4| + e^{7x_2-36} = 2, \quad \log_{10}\left(\frac{12x_1^2}{x_2} - 6\right) + x_1^4 = 9.$$

If we use q-method for $q = 0.9$, we yield the following iterations for the exact solutions $(x_1, x_2) = (\sqrt{3}, 36/7)$:

$$\vec{x}^{(k)} : \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1.78067 \\ 5.29844 \end{bmatrix}, \begin{bmatrix} 1.73405 \\ 5.20213 \end{bmatrix}, \begin{bmatrix} 1.73208 \\ 5.15274 \end{bmatrix}, \begin{bmatrix} 1.73205 \\ 5.14302 \end{bmatrix} \to \begin{bmatrix} 1.73205 \\ 5.14286 \end{bmatrix}.$$

The classical Newton–Kantorovich method with initial values $x_1 = 2$, $x_2 = 5$ can not be used in this case because the partial derivative of the first function with respect to the first variable does not exist.

Example 7.4. For known q ($0 < |q| < 1$), the solutions x and y of the system with some infinite products

$$\prod_{n=1}^{\infty}(1 - xq^n) \Big/ \prod_{n=1}^{\infty}(1 - yq^n) = 1/2$$

$$\prod_{n=1}^{\infty}(1 - xq^n) + e^{\prod_{n=1}^{\infty}(1-yq^n)} = 5$$

can be also found by this method. For example, for $q = 0.75$, we have the solutions $x = 0.104199, \ldots$ and $y = -0.127765, \ldots$.

Remark. Another approach to this system is to solve the system introducing new notation for products, and then, to find x and y by our q-method from the fifth section.

Example 7.5. Let us consider the equation

$$\int_0^x g(t)\,\mathrm{d}_{3/4}t - \frac{64\sqrt[4]{8}}{128 - 27\sqrt{3}} = 0.$$

Applying q-Newton method, we get

$$x_0 = 1.32499, x_1 = 1.45871, x_2 = 1.39966, x_3 = 1.41999,$$
$$x_4 = 1.41207, \ldots, x_{10} = 1.41421.$$

Really, the exact solution is $x = \sqrt{2}$ and the function is $g(t) = t^{5/2}$.

Acknowledgement

Supported by Ministry of Sci. & Techn. Rep. Serbia, the projects no. 1409/1379.

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