The Sobolev orthogonal polynomials in computing of the Hankel determinants

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Abstract. In this paper, we study closed form evaluation for some special Hankel determinants arising in combinatorial analysis, especially, for the bidirectional number sequences. We show that such problems are directly connected with the theory of quasi-definite discrete Sobolev orthogonal polynomials. It opens a lot of procedural dilemmas which we will try to exceed. A few examples deal with Fibonacci numbers and power sequences will illustrate our considerations. We believe that our usage of orthogonality on Sobolev spaces is quite new.

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1 Introduction

The *Hankel transform* of a given number sequence A is the sequence of Hankel determinants H given by

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \to \quad H = \{h_n\}_{n\in\mathbb{N}} : \quad h_n = |a_{i+j-2}|_{i,j=1}^{n}.$$

The closed-form computation of Hankel determinants are of great combinatorial interest related to partitions and permutations. A lot of methods is known for evaluation of these determinants a long list of known determinant evaluations can be seen in [11]. A few papers deal with special number sequences and their Hankel determinants [6] and [7].

Between the methods for evaluating the Hankel determinants, our attention occupies the method based on the theory of distributions and orthogonal polynomials. We have published our considerations about them in the papers [9], [13] and [4].

Namely, the Hankel determinant h_n of the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1} \qquad (n = 1, 2, \ldots) \ , \tag{1}$$

where the sequence $\{\beta_n\}_{n\in\mathbb{N}_0}$ is the sequence of the coefficients in the recurrence relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x) \ , \qquad P_{-1} \equiv 0, \quad P_0 \equiv 1. \tag{2}$$

Here, $\{P_n(x)\}_{n\in\mathbb{N}_0}$ is the monic polynomial sequence orthogonal with respect to the inner product

$$(f,\ g) = \mathcal{U}[f(x)g(x)],$$

where $\mathcal{U}:\mathcal{P}\to\mathbb{R}$ ($\mathcal{P}$ is a space of one-variable polynomials with real coefficients) is a linear real quasi-definite functional determined by

$$a_n = \mathcal{U}[x^n] \quad (n = 0, 1, 2, \ldots) \ .$$

In some cases, there exists weight function $w(x)$ such that the functional $\mathcal{U}$ can be represented by

$$\mathcal{U}[f] = \int_{\mathbb{R}} f(x)\ w(x)\ dx \qquad \big(f\in\mathcal{P};\ w(x)\geq 0\big) \ .$$

So, we can join to every weight $w(x)$ two sequences of coefficients, i.e.

$$w(x) \ \mapsto\ \{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0} \ ,$$

by

$$\alpha_n = \frac{\mathcal{U}[xP_n^2(x)]}{\mathcal{U}[P_n^2(x)]} \ , \qquad \beta_n = \frac{\mathcal{U}[P_n^2(x)]}{\mathcal{U}[P_{n-1}^2(x)]} \qquad (n\in\mathbb{N}_0) \ .$$

The statements of the next lemma are very useful (see proofs, for example, in [10]).

Lemma 1. *Let $w(x)$ be a weight function with the support $supp(w) = (a, b)$ and $\{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}$ the corresponding sequences of coefficients in monic three-term recurrence relation. Also, let $\tilde{w}(x)$ be a modified weight and $\{\tilde{\alpha}_n,\ \tilde{\beta}_n\}_{n\in\mathbb{N}_0}$ its sequences of coefficients. It is valid:*

(i) *if* $\tilde{w}(x) = c_1 w(x) \quad\Rightarrow\quad \tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = c_1\beta_0, \quad \tilde{\beta}_n = \beta_n\ (n\in\mathbb{N})$;

(ii) *if* $\tilde{w}(x) = w(c_1 x + d_1)$ *then*

$$\tilde{\alpha}_n = \frac{\alpha_n - d_1}{c_1},\ \tilde{\beta}_0 = \frac{\beta_0}{|c_1|},\ \tilde{\beta}_n = \frac{\beta_n}{c_1^2}\ (n\in\mathbb{N}),\ \mathrm{supp}(\tilde{w}) = \Big(\frac{a-d_1}{c_1}, \frac{b-d_1}{c_1}\Big);$$

(iii) *if* $\tilde{w}_{c_1}(x) = (x - c_1)\ w(x)$ *such that* $c_1 < a < b$ *then*

$$\begin{aligned} \tilde{\alpha}_{c_1,0} &= \alpha_0 + r_1 - r_0 \ , & \tilde{\alpha}_{c_1,n} &= \alpha_{n+1} + r_{n+1} - r_n, \\ \tilde{\beta}_{c_1,0} &= -r_0\beta_0, & \tilde{\beta}_{c_1,n} &= \beta_n\ \frac{r_n}{r_{n-1}} & (n\in\mathbb{N}) \ , \end{aligned}$$

where

$$r_0 = c_1 - \alpha_0, \quad r_{n+1} = c_1 - \alpha_{n+1} - \frac{\beta_{n+1}}{r_n} \quad (n\in\mathbb{N}_0);$$

(iv) *if* $\tilde{w}_{d_1}(x) = (d_1 - x)\ w(x)$ *such that* $a < b < d_1$ *then*

$$\tilde{\alpha}_{d_1,n} = d_1 + q_{n+1} + e_{n+1}\ (n\in\mathbb{N}_0), \quad \tilde{\beta}_{d_1,0} = \int_{\mathbb{R}} \tilde{w}_{d_1}(x)\ dx, \quad \tilde{\beta}_{d_1,n} = q_{n+1}e_n \quad (n\in\mathbb{N}) \ ,$$

where $e_0 = 0, \quad q_n = \alpha_{n-1} - e_{n-1} - d_1, \quad e_n = \beta_n/q_n \quad (n\in\mathbb{N});$

(v) *if*

$$\tilde{w}_{c_1}(x) = \frac{w(x)}{x - c_1} \qquad (c_1 < a < b) \ ,$$

then

$$\begin{aligned} \tilde{\alpha}_{c_1,0} &= \alpha_0 + r_0 \ , & \tilde{\alpha}_{c_1,n} &= \alpha_n + r_n - r_{n-1}, \\ \tilde{\beta}_{c_1,0} &= -r_{-1}, & \tilde{\beta}_{c_1,n} &= \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \qquad (n \in \mathbb{N}) \ , \end{aligned}$$

where

$$r_{-1} = -\int_{\mathbb{R}} \tilde{w}_{c_1}(x)\ dx, \qquad r_n = c_1 - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0);$$

(vi) *if*

$$\tilde{w}_{d_1}(x) = \frac{w(x)}{d_1 - x} \qquad (a < b < d_1) \ ,$$

then

$$\begin{aligned} \tilde{\alpha}_{d_1,0} &= \alpha_0 + r_0 \ , & \tilde{\alpha}_{d_1,n} &= \alpha_n + r_n - r_{n-1}, \\ \tilde{\beta}_{d_1,0} &= r_{-1}, & \tilde{\beta}_{d_1,n} &= \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \qquad (n \in \mathbb{N}) \ , \end{aligned}$$

where

$$r_{-1} = \int_{\mathbb{R}} \tilde{w}_{d_1}(x)\ dx, \qquad r_n = d_1 - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0) \ .$$

Corollary 2. *The coefficients in three term recurrence relation for polynomials orthogonal with respect to the inner product defined by*

$$w^{(2)}(x) = \frac{1}{2} \cdot \sqrt{\frac{2-x}{2+x}}, \quad x \in (-2, 2),$$

are given by

$$\alpha_0^{(2)} = 1, \quad \alpha_n^{(2)} = 0; \qquad \beta_0^{(2)} = \pi, \qquad \beta_n^{(2)} = 1 \quad (n \in \mathbb{N}). \tag{3}$$

Proof. Let us start with the monic orthogonal polynomials $\{V_n(x)\}_{n\in\mathbb{N}_0}$ with respect to the

$$w^*(x) = p^{(1/2,-1/2)}(x) = \sqrt{\frac{1-x}{1+x}}, \qquad x \in (-1, 1).$$

These polynomials are monic Chebyshev polynomials of the fourth kind and they can be expressed (see Szegö [14]) by

$$V_n(\cos\theta) = \frac{\sin(n + \frac{1}{2})\theta}{2^n \sin\frac{1}{2}\theta}.$$

They satisfy the three-term recurrence relation (see, for example [8]):

$$V_{n+1}(x) = (x - \alpha_n^*)\ V_n(x) - \beta_n^* V_{n-1}(x) \quad (n = 0, 1, \ldots),$$

with initial values $V_{-1}(x) = 0, \qquad V_0(x) = 1$, where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0; \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \quad (n \in \mathbb{N}).$$

Let

$$w^{(1)}(x) = w^*\Big(-\frac{1}{2}\ x\Big) = \sqrt{\frac{2+x}{2-x}}, \quad x \in (-2,2).$$

By Lemma 1, case (ii), we have

$$\alpha_n^{(1)} = 2\alpha_n^* = 0 \quad (n \in \mathbb{N}_0), \qquad \beta_0^{(1)} = 2\beta_0^* = \pi, \qquad \beta_n^{(1)} = 4\beta_n^* = 1 \quad (n \in \mathbb{N}).$$

At last, considering the weight

$$w^{(2)}(x) = \frac{1}{2} \cdot w^{(1)}(x), \quad x \in (-2,2),$$

by Lemma 1, case (i), we find coefficients (3). □

2 The quasi-definite case of discrete Sobolev inner product

Until now, orthogonality on Sobolev spaces was considered because of the spectral theory of ordinary and partial differential equations and the numerical methods for their solving, and expansions of the functions in the special Fourier series (see [1], [2] and [12]).

We consider polynomial sequence orthogonal with respect to a discrete Sobolev inner product, that is, an ordinary inner product on the real line plus an atomic inner product. So introduced Sobolev orthogonal polynomials we research starting with its classical counterpart including the influence of the addition of function value in a special point.

According to our knowledge, it was not noted their application in the computing of any Hankel determinant.

Let $(\cdot,\cdot)$ be a positive definite inner product and $\{P_n(x)\}_{n\in\mathbb{N}_0}$ the corresponding monic orthogonal polynomial sequence which satisfies three-term recurrence relation (2).

The sequence of monic polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ orthogonal with respect to the discrete Sobolev inner product

$$\langle f,\ g\rangle = A(f,\ g)\ +\ Bf(c)g(c) \qquad (A, B, c \in \mathbb{R}), \tag{4}$$

is quite determined by $\{P_n(x)\}_{n\in\mathbb{N}_0}$ and constants $A,\ B$ and c. If we take $B = 0$ we get standard continuous case, while taking $A = 0$ we get discrete case. F. Marcellan and A. Ronveaux in [12] researched the case $A = 1$ and $B = 1/\lambda > 0$, where λ is a parameter. In the present paper, we let A and B to be any real numbers which assure existence of the sequence of Sobolev orthogonal polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$.

Since $\{P_k(x)\}_{0\leq k\leq n}$ is a basis in the subspace $\mathcal{P}_n$ of all polynomials whose degree do not exceed n, we can write

$$Q_n(x) = P_n(x)\ +\ \sum_{j=0}^{n-1} \frac{(Q_n, P_j)}{(P_j, P_j)} P_j(x)\ .$$

Based on the orthogonality, we can write

$$0 = \langle Q_n,\ P_j \rangle = A(Q_n, P_j) + BQ_n(c)P_j(c),$$

wherefrom

$$Q_n(x) = P_n(x)\ - \frac{B}{A}\ Q_n(c) \sum_{j=0}^{n-1} \frac{P_j(c)P_j(x)}{(P_j, P_j)}\ .$$

By introducing the kernel

$$K_n(x, y) = \sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{(P_j, P_j)}\ ,$$

we can write

$$Q_n(x) = P_n(x)\ - \frac{B}{A}\ Q_n(c)\ K_{n-1}(x, c)\ . \tag{5}$$

Putting $x = c$ in the previous relation, we get

$$Q_n(c) = P_n(c)\ - \frac{B}{A}\ Q_n(c)K_{n-1}(c, c) \quad \Rightarrow \quad Q_n(c) = \frac{AP_n(c)}{A + BK_{n-1}(c, c)}\ .$$

By applying (5), for n and $n+1$, after some computation, we have

$$\begin{vmatrix} Q_n(x) & Q_{n+1}(x) \\ Q_n(c) & Q_{n+1}(c) \end{vmatrix} = \frac{A}{A + BK_{n-1}(c, c)}\ \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_n(c) & P_{n+1}(c) \end{vmatrix}\ .$$

Also, multiplying (5) with $(x - c)$ and using three-term recurrence relation for $\{P_n(x)\}_{n\in\mathbb{N}_0}$, we yield

$$(x - c)Q_n(x) = P_{n+1}(x) + \big(\alpha_n - c - \rho_n P_{n-1}(c)\big)P_n(x) + \rho_n P_n(c)P_{n-1}(x)\ ,$$

where

$$\rho_n = \frac{B}{(P_{n-1}, P_{n-1})} \cdot \frac{P_n(c)}{A + BK_{n-1}(c, c)} \qquad (n \in \mathbb{N}).$$

Directly from the orthogonality in discrete Sobolev space it follows

$$(x - c)Q_n(x) = Q_{n+1}(x) + e_{n,n}Q_n(x) + e_{n,n-1}Q_{n-1}(x), \tag{6}$$

where $e_{n,n}$ and $e_{n,n-1}$ are some real constants.

Theorem 3. *The polynomial sequence $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ orthogonal with respect to the discrete Sobolev inner product (4) satisfy three-term recurrence relation of the form:*

$$Q_{n+1}(x) = (x - \sigma_n)Q_n(x) - \tau_n Q_{n-1}(x) \quad (n \in \mathbb{N}), \qquad Q_{-1} \equiv 0,\ Q_0 \equiv 1, \tag{7}$$

where

$$\sigma_n = \alpha_n + \rho_{n+1}P_n(c) - \rho_n P_{n-1}(c),$$
$$\tau_n = \frac{\rho_n P_n(c) + \beta_n}{\rho_{n-1}P_{n-1}(c) + \beta_{n-1}} \qquad (n \in \mathbb{N}).$$

Here $\tau_0 = \langle 1, 1\rangle$.

Proof. The recurrence relation (6) can be rearranged in the form (7). By multiplying it with $(x - c)$ and applying (5), from linear independence of the sequence $\{P_n(x)\}_{n\in\mathbb{N}_0}$ we find σ_n and τ_n. □

3 The Hankel determinants of a special integer sequence

We are interested in investigating sequences that are defined as the image of bidirectional sequences. Such sequences have been studied, for instance, by Basor and Ehrhardt [5]. They arise naturally in the Fourier analysis of signals and systems. The authors in are interested in deriving asymptotic behavior of the Hankel determinants which are important in statistical mechanics, random matrix theory and the theory of orthogonal polynomials.

3.1 The Fibonacci case

Let

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (a_{n-2k+1} + a_{n-2k}), \quad \text{where} \quad a_n = F_{|n|}. \tag{8}$$

Here $F_{|n|}$ is $|n|$-th Fibonacci number:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \in \mathbb{N}).$$

The first members of $\{b_n\}_{n\in\mathbb{N}_0}$ are $\{1, 3, 7, 17, 39, \ldots\}$. P. Barry and A. Hennessy in [3] have proved that the generating function of the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ is

$$\sum_{n=0}^{\infty} b_n x^n = \frac{1+2x}{1-5x^2}\left(1 + \frac{x}{\sqrt{1-4x^2}}\right). \tag{9}$$

Lemma 4. *The sequence $\{b_n\}_{n\in\mathbb{N}_0}$ satisfies the next recurrence relation*

$$b_n - 5b_{n-2} = \begin{cases} 2^{(n-1)/2} \frac{(n-2)!!}{((n-1)/2)!}, & n - \text{odd}, \\ \\ 2^{n/2} \frac{(n-3)!!}{(n/2-1)!}, & n - \text{even}. \end{cases} \tag{10}$$

Proof. Directly from the expression (9), we have

$$\begin{aligned} \sum_{n=2}^{\infty} (b_n - 5b_{n-2}) x^n &= \frac{x(1+2x)}{\sqrt{1-4x^2}} - x = -x + (x+2x^2) \sum_{n=0}^{\infty} \binom{-1/2}{n} (-4)^n x^{2n} \\ &= \sum_{n=2}^{\infty} \binom{-1/2}{n-1} (-4)^{n-1} x^{2n-1} + 2 \sum_{n=1}^{\infty} \binom{-1/2}{n-1} (-4)^{n-1} x^{2n} \\ &= \sum_{n=2}^{\infty} 2^{n-1} \frac{(2n-3)!!}{(n-1)!} x^{2n-1} + \sum_{n=1}^{\infty} 2^n \frac{(2n-3)!!}{(n-1)!} x^{2n}. \end{aligned}$$

□

Lemma 5. *The next moment representation is valid*

$$b_n = -\frac{1}{\pi} \int_{-2}^{2} \frac{x^n}{c^2 - x^2} \sqrt{\frac{2+x}{2-x}}\, dx + \left(1 + \frac{2}{c}\right) c^n \qquad \left(c = \sqrt{5}\right) (n \in \mathbb{N}). \tag{11}$$

Proof. Let us denote by B_n the right side of (11). Then

$$B_n - 5B_{n-2} = -\frac{1}{\pi}\int_{-2}^{2} \frac{x^n - 5x^{n-2}}{c^2 - x^2}\sqrt{\frac{2+x}{2-x}}\, dx + \left(1 + \frac{2}{c}\right)(c^n - 5c^{n-2}),$$

wherefrom

$$B_{2n} - 5B_{2n-2} = \frac{1}{\pi}\int_{-2}^{2} x^{2n-2}\sqrt{\frac{2+x}{2-x}}\, dx = \frac{2^{2n-1}\Gamma(n-1/2)}{\sqrt{\pi}\Gamma(n)}\ ,$$

and

$$B_{2n-1} - 5B_{2n-3} = \frac{1}{\pi}\int_{-2}^{2} x^{2n-3}\sqrt{\frac{2+x}{2-x}}\, dx = \frac{2^{2n}\Gamma(n-1/2)}{\sqrt{\pi}\Gamma(n)}\ .$$

It proves that both sequences $\{B_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ satisfies the same recurrence relation (10). To prove that $B_n = b_n$ for every $n \in \mathbb{N}_0$, we just need to show that initial terms B_0 and b_0 are equal. That can be done by direct evaluation of both terms:

$$B_0 = -\frac{1}{\pi}\int_{-2}^{2} \frac{1}{c^2 - x^2}\sqrt{\frac{2+x}{2-x}}\, dx + \left(1 + \frac{2}{c}\right) = 1 = b_0.$$

That completes the proof of the lemma. □

Note that (11) is the moment sequence corresponding to the special case of discrete Sobolev inner product (4) for $A = -1$ and $B = 1 + \frac{2}{\sqrt{5}}$:

$$\langle f,\ g\rangle = -(f,\ g)\ +\ \left(1 + \frac{2}{\sqrt{5}}\right) f(c)g(c) \qquad (A, B, c \in \mathbb{R}) \tag{12}$$

with

$$(f,\ g) = \int_{-2}^{2} f(x)g(x)w(x)\, dx, \tag{13}$$

where

$$w(x) = \frac{1}{\pi(5 - x^2)}\sqrt{\frac{2+x}{2-x}}, \quad x \in (-2, 2), \tag{14}$$

Our goal is to evaluate corresponding coefficients τ_n which will lead to the expression for the Hankel transform of the sequence b_n. Evaluation will be performed in two steps. The first step is to find expressions for the coefficients α_n and β_n corresponding to the weight function $w(x)$ (i.e. the absolute continuous part $(\cdot,\cdot)$ of (12)). It is done in the following lemma.

Lemma 6. *The coefficients α_n and β_n of the three-term recurrence relation, corresponding to the weight function (14), are given by:*

$$\begin{aligned} \alpha_0 &= \frac{5-\sqrt{5}}{2}\ , \quad \alpha_1 = \frac{\sqrt{5}-3}{2}\ , \quad \alpha_n = 0; \\ \beta_0 &= \frac{2}{\sqrt{5}}\ , \quad \beta_1 = \frac{3\sqrt{5}-5}{2}\ , \quad \beta_n = 1 \quad (n \geq 2). \end{aligned} \tag{15}$$

Proof. We will continue the considerations started in Corollary 2. Let

$$w^{(3)}(x) = \frac{w^{(2)}(x)}{\sqrt{5}-x}, \qquad x \in (-2,2).$$

Applying Lemma 1 (vi), for $d_1 = \sqrt{5}$, we have $r_{-1} = \pi(1+\sqrt{5})/2,\ r_k = (\sqrt{5}-1)/2\ (k \in \mathbb{N}_0)$. Hence

$$\alpha_0^{(3)} = \frac{1+\sqrt{5}}{2},\ \alpha_n^{(3)} = 0\ (n \in \mathbb{N});\quad \beta_0^{(3)} = \frac{\pi}{2}(1+\sqrt{5}),\ \beta_1^{(3)} = \frac{3-\sqrt{5}}{2},\ \beta_n^{(3)} = 1\ (n \geq 2).$$

Similarly, let

$$w^{(4)}(x) = \frac{w^{(3)}(x)}{x-(-\sqrt{5})}, \quad x \in (-2,2).$$

According to Lemma 1 (v), for $c_1 = -\sqrt{5}$, we have $r_{-1} = -\pi/\sqrt{5},\ r_0 = 2-\sqrt{5},\ r_k = (1-\sqrt{5})/2\ (k \in \mathbb{N})$. Hence

$$\begin{aligned} \alpha_0^{(4)} &= \frac{5-\sqrt{5}}{2}, \quad & \alpha_1^{(4)} &= \frac{\sqrt{5}-3}{2}, \quad & \alpha_n^{(4)} &= 0; \\ \beta_0^{(4)} &= \frac{\pi}{\sqrt{5}}, & \beta_1^{(4)} &= \frac{3\sqrt{5}-5}{2}, & \beta_n^{(4)} &= 1 \qquad (n = 2,3,\ldots). \end{aligned}$$

Finally the last transformation

$$w(x) = \frac{2}{\pi}\, w^{(4)}(x),$$

directly implies (15). □

Lemma 7. *The coefficients* $\{\tau_n\}$ *of the three-term recurrence relation, satisfied by Sobolev orthogonal polynomials* $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ *orthogonal with respect to* (12), *are given by:*

$$\tau_0 = 1, \quad \tau_1 = -2, \quad \tau_n = 1 \quad (n \geq 2).$$

Proof. Denote by $\{P_n(x)\}_{n\in\mathbb{N}_0}$ the sequence of monic orthogonal polynomials with respect to the weight function $w(x)$ (i.e. absolute continuous scalar product (13)). Their squared norms are

$$(P_0,\ P_0) = \frac{2}{\sqrt{5}}, \qquad (P_n,\ P_n) = \beta_n\beta_{n-1}\ldots\beta_0 = 3-\sqrt{5} \qquad (n \in \mathbb{N})\ .$$

By mathematical induction, we can prove that the polynomials $\{P_n(x)\}_{n\in\mathbb{N}_0}$ at the point $c = \sqrt{5}$ have the following values:

$$P_0(c) = 1, \quad P_1(c) = \frac{3\sqrt{5}-5}{2}\ , \quad P_n(c) = (5-2\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n \quad (n = 2,3,\ldots)\ .$$

Let us denote by $K_n = K_n(c,c)$ $(n \in \mathbb{N}_0)$. Here, it is

$$K_0 = \frac{\sqrt{5}}{2}, \quad K_1 = \frac{3}{4}(5-\sqrt{5}),$$
$$K_n = \frac{\sqrt{5}+1}{8}\left(14\sqrt{5}-30+5(3\sqrt{5}-7)\left(\frac{1+\sqrt{5}}{2}\right)^{2n}\right) \quad (n = 2,3,\ldots)\ ,$$

and

$$\rho_1 = \frac{1+\sqrt{5}}{2}, \qquad \rho_n = \frac{15+7\sqrt{5}}{10}\left(\frac{\sqrt{5}-1}{2}\right)^n, \quad \rho_n\ P_n(c) = \frac{1+\sqrt{5}}{2} \quad (n = 2,3,\ldots)\ .$$

Hence $\tau_n = 1 \quad (n \in \mathbb{N};\ n \geq 2)$. It is known that $\tau_0 = b_0 = 1$. The first members are

$$Q_0(x) = 1, \quad Q_1(x) = x-3, \quad Q_2(x) = x^2-2x-1 = (x+1)Q_1(x)-(-2)Q_0(x),$$

wherefrom $\tau_1 = -2$. □

Now, we have all elements for formula (1) and we can compute h_n by

$$h_n = \tau_0^n \tau_1^{n-1} \tau_2^{n-2} \cdots \tau_{n-2}^2 \tau_{n-1}\ .$$

That completes the proof of the following theorem:

Theorem 8. *The Hankel transform of* $\{b_n\}_{n\in\mathbb{N}_0}$ *defined by (8) is given by* $h_n = (-2)^{n-1}$ $(n \in \mathbb{N})$.

3.2 A power case

Let

$$b_n = \sum_{k=0}^{n} \binom{n}{k}(a_{n-2k+1}+a_{n-2k}), \quad \text{where} \quad a_n = r^{|n|} \quad (r > 1). \tag{16}$$

The sequence $\{b_n\}_{n\in\mathbb{N}_0}$ has the next generating function (see [3])

$$\sum_{n=0}^{\infty} b_n x^n = \frac{r+1}{r-(r^2+1)x}\left(\frac{r+1}{2} + \frac{(r-1)(x+1/2)}{\sqrt{1-4x^2}}\right).$$

Lemma 9. *The sequence* $\{b_n\}_{n\in\mathbb{N}_0}$ *satisfies the following recurrence relation*

$$\frac{r}{r^2-1}\left(b_n - \frac{r^2+1}{r}b_{n-1}\right) = \begin{cases} 2^{(n-1)/2}\frac{(n-2)!!}{((n-1)/2)!}\ , & n - \text{odd}, \\ 2^{n/2-1}\frac{(n-1)!!}{(n/2)!}, & n - \text{even}; \end{cases} \quad (n \in \mathbb{N}). \tag{17}$$

Lemma 10. *The next moment representation is valid*

$$b_n = \frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{x^n}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx + \frac{(r+1)^2}{r}c^n \qquad \left(c=\frac{r^2+1}{r}\right)\ (n\in\mathbb{N}). \tag{18}$$

Proof. We proceed similarly as in Lemma 5. Let us denote by B_n the right-hand side of (18). Hence,

$$\begin{aligned} B_n - cB_{n-1} &= \frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{x^n - cx^{n-1}}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx + \frac{(r+1)^2}{r}(c^n - c\cdot c^{n-1}) \\ &= \frac{(r^2-1)}{2\pi r}\int_{-2}^{2}x^{n-1}\sqrt{\frac{2+x}{2-x}}\,dx = 2^{n-1}\frac{(r^2-1)}{\pi r}\int_{-1}^{1}t^{n-1}\frac{\sqrt{1-t^2}}{1-t}\,dt. \end{aligned}$$

which implies

$$B_{2m+1} - cB_{2m} = 2^{2m}\frac{(r^2-1)}{\sqrt{\pi}r}\frac{\Gamma(m+1/2)}{m!}, \qquad B_{2m} - cB_{2m-1} = 2^{2m-1}\frac{(r^2-1)}{\sqrt{\pi}r}\frac{\Gamma(m+1/2)}{m!}.$$

That proves that sequences $\{B_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ satisfies the same recurrent relation (17). To prove that $B_n = b_n$ for every $n\in\mathbb{N}_0$, we just need to show that initial terms B_0 and b_0 are equal, which again can be done by direct evaluation:

$$B_0 = \frac{-(r^2-1)}{2\pi r}\int_{-2}^{2}\frac{1}{c-x}\sqrt{\frac{2+x}{2-x}}\,dx + \frac{(r+1)^2}{r} = 1 + r = b_0.$$

That completes the proof of the lemma. □

Therefore, for the further consideration, the important weight function is given by

$$w(x) = \frac{r^2-1}{\pi r}\ w^{(3)}(x) = \frac{r^2-1}{\pi r}\frac{1}{2(c-x)}\sqrt{\frac{2+x}{2-x}}, \quad x\in(-2,2), \tag{19}$$

Lemma 11. *The coefficients α_n and β_n of the three-term recurrence relation, corresponding to the weight function (19), are given by:*

$$\alpha_0 = \frac{r+1}{r}, \quad \alpha_n = 0\ (n\in\mathbb{N}); \qquad \beta_0 = \frac{r+1}{r}, \quad \beta_1 = \frac{r-1}{r},\ \beta_n = 1 \quad (n\ge 2). \tag{20}$$

Proof. Let

$$w^{(3)}(x) = \frac{w^{(2)}(x)}{c-x}, \quad x\in(-2,2).$$

Applying Lemma 1 (vi), for $d_1 = c = (r^2+1)/r$, we have $r_{-1} = \pi/(r-1),\ r_k = 1/r \quad (k\in\mathbb{N}_0)$. Hence

$$\alpha_0^{(3)} = \frac{r+1}{r},\ \alpha_n^{(3)} = 0\ (n\in\mathbb{N}); \quad \beta_0^{(3)} = \frac{\pi}{r-1},\ \beta_1^{(3)} = \frac{r-1}{r},\ \beta_n^{(3)} = 1 \quad (n=2,3,\ldots).$$

Finally, for

$$w(x) = \frac{r^2-1}{\pi r}\ w^{(3)}(x)$$

we get (20). □

Theorem 12. *Let* $a_n = r^{|n|}$, *and* $\{b_n\}_{n\in\mathbb{N}_0}$ *sequence determined by* (16). *Then the Hankel transform of* $\{b_n\}_{n\in\mathbb{N}_0}$ *is given by* $h_n = (r+1)(1-r^2)^{n-1}$ $(n \in \mathbb{N})$.

Proof. The squared norms of polynomials orthogonal polynomials with respect to (19) are

$$(P_0,\ P_0) = \frac{r+1}{r}, \qquad (P_n,\ P_n) = \beta_n\beta_{n-1}\cdots\beta_0 = \frac{r^2-1}{r^2} \qquad (n \in \mathbb{N})\ .$$

By mathematical induction, we prove that these polynomials have the following values in the point c:

$$P_0(c) = 1, \qquad P_n(c) = (r-1)r^{n-1} \qquad (n \in \mathbb{N})\ .$$

Now, it is valid

$$K_m(c,c) = \frac{r(r^{2m+1}+1)}{(r+1)^2} \quad (m \in \mathbb{N}_0), \qquad \rho_n = (r+1)r^{1-n} \quad (n \geq 1).$$

Notice that $\rho_n\ P_n(c) = r^2 - 1$ $(n \geq 2)$. Now, it is $\tau_n = 1$ $(n \geq 2)$.

The first few members of the sequence $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ are:

$$Q_0 = 1, \quad Q_1 = x - r - 1, \quad Q_2 = x^2 - (r+1)x + r - 1 = xQ_1 - (1-r)Q_0.$$

Hence $\tau_0 = b_0 = r+1, \quad \tau_1 = 1 - r$. The Hankel transform of the sequence b_n defined by (16) is

$$h_n = \tau_0^n\tau_1^{n-1}\tau_2^{n-2}\cdots\tau_{n-2}^2 = (r+1)^n(1-r)^{n-1}\ .$$

□

Example 13. Let $r = 2$. Then $a_n = 2^{|n|}$ and

$$b_n = \sum_{k=0}^{n}\binom{n}{k}\left(2^{|n-2k+1|} + 2^{|n-2k|}\right)$$

which begins with

$$3,\ 9,\ 24,\ 63,\ 162,\ 414,\ 1050,\ 2655,\ldots$$

and has the generating function

$$\sum_{n=0}^{\infty} b_n(2)x^n = \frac{3(2-3x-x^2c(x^2))}{(1-2x)(2-5x)}, \quad \text{where} \quad c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

We have the following moment representation

$$b_n = \frac{1}{2\pi}\int_{-2}^{2} x^n \frac{3\sqrt{4-x^2}}{(2-x)(2x-5)}\,dx + \frac{9}{2}\left(\frac{5}{2}\right)^n.$$

Hence $\tau_0 = 3,\ \tau_1 = -1,\ \tau_n = 1 \quad (n \in \mathbb{N};\ n \geq 2)$. The Hankel transform of the sequence b_n is

$$h_n = 3^n(-1)^{n-1} \qquad (n = 1, 2, \ldots).$$

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