An application of Sobolev orthogonal polynomials in the computing of a special Hankel determinant

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Abstract. Many Hankel determinants computations arising in combinatorial analysis, can be done by results from the theory of standard orthogonal polynomials. Here, we will emphasize a special sequence which requires including of discrete Sobolov orthogonality in order to find their closed form.

1 Introduction

The *Hankel transform* of a given number sequence A is the sequence of Hankel determinants H given by

$$A = \{a_n\}_{n\in\mathbb{N}_0} \quad \rightarrow \quad H = \{h_n\}_{n\in\mathbb{N}_0}: \quad h_n = |a_{i+j-2}|_{i,j=1}^{n}. \tag{1}$$

This term is first used by J.W. Layman [7]. There is close connection of evaluations of The closed-form computation of Hankel determinants is of great combinatorial interest related to partitions and permutations. A lot of methods is known for evaluation of these determinants a long list of known determinant evaluations can be seen in [6]. We also found a few interesting results exposed in [3] and [9].

Here, we will consider the sequence A whose arbitrary element is

$$a_n = \sum_{k=0}^{n} T_{n,k} r^k \quad (r \in \mathbb{Z}), \tag{2}$$

where

$$T_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1}\binom{2n}{n-k} \quad (0 \le k \le n;\ n \in \mathbb{N}_0). \tag{3}$$

We will imply that $T_{n,k} = 0$ in other cases.

The first members of A and its Hankel transform are:

$$A = \left\{1, 1+r, 2+3r+r^2, 5+9r+5r^2+r^3, \ldots\right\}, \quad H = \left\{1, 1+r, (1+r)^2, (1+r)^3, \ldots\right\}.$$

Our purpose is to prove that, in general, it is valid $h_n = (1+r)^{n-1}$.

2 Preliminaries

The ordinary generating function of the sequence a_n is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad \text{with} \qquad a_n = [x^n] f(x), \tag{4}$$

where the operator $[x^n]$ extracts the coefficient of x^n. Sequences are often referred to by their 'A' number in the On-Line Encyclopedia of Integer Sequences [10].

Let us remand that the sequence of Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad (n \in \mathbb{N}_0) \tag{5}$$

can be written in the integral form

$$C_n = \frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}} dx \qquad (n \in \mathbb{N}_0) \tag{6}$$

This sequence has the generating function

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \ . \tag{7}$$

Definition 1. The *Riordan array* is an infinite lower-triangular matrix $\mathcal{M} = [m_{j,k}]$ defined by a pair of generating functions $g(x) = \sum_{k=0}^{\infty} g_k x^k$ and $f(x) = \sum_{k=1}^{\infty} f_k x^k$ where $f_1 \neq 0$, whose k-th column is generated by $g(x)f(x)^k$ (the first column being indexed by 0), i.e.

$$m_{j,k} = [x^j]\big(g(x)f(x)^k\big) \qquad (j,k \in \mathbb{N}_0). \tag{8}$$

The matrix $\mathcal{M}$ corresponding to the pair of functions f and g is denoted by (g,f) or $\mathcal{R}(g,f)$.

The *Riordan group* is a set of infinite lower-triangular matrices, where each matrix is defined by a pair of generating functions.

We will consider the transform defined by the Riordan array

$$\big(c(x), xc^2(x)\big). \tag{9}$$

This matrix is the inverse of the Riordan matrix $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$, which is the coefficient array of the Morgan Voyce polynomials. They are closely linked to the Chebyshev polynomials of the second kind.

The general element $T_{n,k}$ of $(c(x), xc^2(x))$ is given by (3). This is the sequence A039599 and represents the number of lattice paths from $(0,0)$ to (n,n) with steps $E=(1,0)$ and $N=(0,1)$ which touch but do not cross the line $x-y=k$ and only situated above this line. Also, This sequence appears in the definition of the "Ballot" transform (see [1]).

By the theory of Riordan arrays, the generating function of a_n can be evaluated from

$$\big(c(x), xc^2(x)\big) \cdot \frac{1}{1-rx} = \frac{c(x)}{1-rxc^2(x)} \ ,$$

i.e.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{(r+1)\ \sqrt{1-4x} + r - 1}{2r - (r+1)^2 x} \ . \tag{10}$$

The function

$$F(x) = \tfrac{1}{x}\ G\big(\tfrac{1}{x}\big) = \frac{(r+1)\ \sqrt{1-\frac{4}{x}} + r - 1}{2rx - (r+1)^2} \tag{11}$$

will be very useful in the next sections.

3 The properties of number sequences

Notice

$$T_{n,0} = C_n, \qquad T_{n,n} = 1, \qquad T_{n,k} = 0 \quad (k < 0 \vee k > n). \tag{12}$$

The sequence $\{T_{n,k}\}$ satisfies the next recurrence relation

$$T_{n,k} = T_{n-1,k-1} + 2T_{n-1,k} + T_{n-1,k+1} \qquad (k = 0, 1, \ldots, n). \tag{13}$$

Let us denote

$$\zeta = \frac{(r+1)^2}{r} \ . \tag{14}$$

Lemma 2. *For the sequence $\{a_n\}$, the next recurrence relation is valid*

$$a_n = \zeta\ a_{n-1} - \frac{r+1}{r} C_{n-1} \ . \tag{15}$$

Proof. We can write

$$\begin{aligned}
a_n - \frac{(r+1)^2}{r}\ a_{n-1} &= a_n - r\ a_{n-1} - 2\ a_{n-1} - \frac{1}{r}\ a_{n-1} \\
&= \sum_{k=0}^{n} T_{n,k} r^k - \sum_{k=0}^{n-1} T_{n-1,k} r^{k+1} - 2 \sum_{k=0}^{n-1} T_{n-1,k} r^k - \sum_{k=0}^{n-1} T_{n-1,k} r^{k-1} \\
&= -\frac{1}{r} T_{n-1,0} + \big(T_{n,0} - 2T_{n-1,0} - T_{n-1,1}\big) \\
&\quad + \sum_{k=1}^{n-2} \big(T_{n,k} - T_{n-1,k-1} - 2T_{n-1,k} - T_{n-1,k+1}\big) r^k \\
&\quad + \big(T_{n,n-1} - 2T_{n-1,n-1} - T_{n-1,n-2}\big) r^{n-1} + \big(T_{n,n} - T_{n-1,n-1}\big) r^n \ .
\end{aligned}$$

By using (12) and (13), we finish the proof. □

Theorem 3. *The sequence $\{a_n\}$ can be represented in the integral form*

$$a_n = \frac{r+1}{2\pi}\int_0^4 \frac{x^n}{(r+1)^2 - rx}\sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r}\ \zeta^n. \tag{16}$$

Proof. We will apply the mathematical induction. Since

$$\frac{r+1}{2\pi}\int_0^4 \frac{1}{(r+1)^2 - rx}\sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r} = 1,$$

and $a_0 = 1$, we have that the formula is valid for $n = 0$.

Suppose that the formula is true for n. Hence

$$\zeta\ a_n = \frac{r+1}{2\pi}\int_0^4 \frac{\zeta x^n}{(r+1)^2 - rx}\sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r}\ \zeta^{n+1}.$$

Since

$$\frac{r+1}{r}C_n = \frac{r+1}{2\pi r}\int_0^4 x^n\ \sqrt{\frac{4-x}{x}}dx,$$

we can write

$$\zeta\ a_n - \frac{r+1}{r}C_n = \frac{r+1}{2\pi}\int_0^4 x^n\Big(\frac{\zeta}{(r+1)^2 - rx} - \frac{1}{r}\Big)\ \sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r}\ \zeta^{n+1}.$$

By using (15), we get

$$a_{n+1} = \frac{r+1}{2\pi}\int_0^4 \frac{x^{n+1}}{(r+1)^2 - rx}\ \sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r}\ \zeta^{n+1}\ ,$$

wherefrom we complete the proof. □

4 The Hankel determinants and orthogonal polynomials

Between a few methods for evaluating the Hankel determinants, our attention occupy the method based on the theory of distributions and orthogonal polynomials.

Namely, the Hankel determinant h_n of the sequence $\{a_n\}_{n\geq 0}$ equals

$$h_n = a_0^n\beta_1^{n-1}\beta_2^{n-2}\cdots\beta_{n-2}^2\beta_{n-1}\ , \tag{17}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cdots}}}\ . \tag{18}$$

The previous sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \beta_n Q_{n-1}(x)\ , \tag{19}$$

where $\{Q_n(x)\}_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{U}$ determined by

$$a_n = \mathcal{U}[x^n] \quad (n = 0, 1, 2, \ldots)\ . \tag{20}$$

In some cases, it exists weight function $w(x)$ such that the functional $\mathcal{U}$ can be expressed by

$$\mathcal{U}[f] = \int_{\mathbb{R}} f(x)\ w(x)\ dx \qquad \big(f(x) \in C(\mathbb{R});\ w(x) \geq 0\big)\ . \tag{21}$$

So, we can join to every weight $w(x)$ two sequences of coefficients, i.e.

$$w(x)\ \mapsto\ \{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}\ , \tag{22}$$

by

$$\alpha_n = \frac{\mathcal{U}[x\ Q_n^2(x)]}{\mathcal{U}[Q_n^2(x)]}\ , \qquad \beta_n = \frac{\mathcal{U}[Q_n^2(x)]}{\mathcal{U}[Q_{n-1}^2(x)]} \qquad (n \in \mathbb{N}_0)\ . \tag{23}$$

Finding of the weight function can be started by the function

$$F(z) = \tfrac{1}{z}\ G\big(\tfrac{1}{z}\big)\ . \tag{24}$$

From the theory of distribution functions (see Chihara [2]), we have Stieltjes inversion function

$$\psi(t) - \psi(s) = -\frac{1}{\pi}\int_s^t \Im F(x+iy)dx. \tag{25}$$

hence we find the distribution function $\psi(t)$. After differentiation of $\psi(t)$ and simplification of the resulting expression, we finally have $w(x) = \psi'(x)$.

The following lemma will be very useful in further discussion.

Lemma 4. *Let*

$$w(x)\ \mapsto\ \{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}, \qquad \tilde{w}(x)\ \mapsto\ \{\tilde{\alpha}_n,\quad \tilde{\beta}_n\}_{n\in\mathbb{N}_0}\ . \tag{26}$$

Then

$$\textit{(i)} \qquad \tilde{w}(x) = Cw(x) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = C\beta_0, \quad \tilde{\beta}_n = \beta_n\ (n \in \mathbb{N})\}\ ; \tag{27}$$

$$\textit{(ii)} \qquad \tilde{w}(x) = w(ax+b) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \frac{\alpha_n - b}{a}, \quad \tilde{\beta}_0 = \frac{\beta_0}{|a|}, \quad \tilde{\beta}_n = \frac{\beta_n}{a^2}\ (n \in \mathbb{N})\}\ ; \tag{28}$$

(iii) If

$$w_c(x) = \frac{\tilde{w}(x)}{x-c} \qquad (c \notin supp(\tilde{w}))\ , \tag{29}$$

then

$$\begin{aligned} \alpha_{c,0} &= \tilde{\alpha}_0 + r_0 \ , & \alpha_{c,k} = &\ \ \tilde{\alpha}_k + r_k - r_{k-1}, \\ \beta_{c,0} &= -r_{-1}, & \beta_{c,k} = &\ \ \tilde{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}) \ , \end{aligned} \tag{30}$$

where

$$r_{-1} = -\int_{\mathbb{R}} w_c(x)\ dx, \qquad r_n = c - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots) \ . \tag{31}$$

(iv) If

$$\hat{w}_d(x) = \frac{\tilde{w}(x)}{d - x} \qquad (d > x,\ \forall x \in supp(\tilde{w})) \ , \tag{32}$$

then

$$\begin{aligned} \hat{\alpha}_{d,0} &= \tilde{\alpha}_0 + r_0 \ , & \hat{\alpha}_{d,k} = &\ \ \tilde{\alpha}_k + r_k - r_{k-1}, \\ \hat{\beta}_{d,0} &= r_{-1}, & \hat{\beta}_{d,k} = &\ \ \tilde{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}) \ , \end{aligned} \tag{33}$$

where

$$r_{-1} = \int_{\mathbb{R}} \hat{w}_d(x)\ dx, \qquad r_n = d - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots) \ . \tag{34}$$

5 The connections of our main problem with classical orthogonal polynomials

From the formula (16), we conclude that the computation of Hankel determinants is directly connected with the monic polynomial sequence $\{Q_n(x)\}$ which is orthogonal with respect to the discrete Sobolev inner product

$$\tilde{\varphi}(f, g) = \frac{r+1}{2\pi}\int_0^4 \frac{f(x)g(x)}{(r+1)^2 - rx}\sqrt{\frac{4-x}{x}}dx + \frac{r-1}{r}f(\zeta)g(\zeta) \qquad \left(\zeta = \frac{(r+1)^2}{r}\right) \tag{35}$$

We will start with a special Jacobi polynomial $P_n(x) = P_n^{(1/2,-1/2)}(x)$ $(n \in \mathbb{N}_0)$, which is also known as *the Chebyshev polynomial of the fourth kind.* The sequence of these polynomials is orthogonal with respect to

$$w^*(x) = w^{(1/2,-1/2)}(x) = \sqrt{\frac{1-x}{1+x}} \ , \qquad x \in (-1, 1) \ .$$

They satisfy the three-term recurrence relation (Chihara [2]):

$$P_{n+1}(x) = (x - \alpha_n^*)\ P_n(x) - \beta_n^* P_{n-1}(x) \quad (n \in \mathbb{N}_0), \qquad P_{-1}(x) = 0 \ , \quad P_0(x) = 1,$$

where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0 \ , \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \qquad (n \in \mathbb{N}) \ .$$

For the weight function

$$\tilde{w}(x) = w\left(\frac{x}{2} - 1\right) = \sqrt{\frac{4-x}{x}} \ , \qquad x \in (0, 4) \ , \tag{36}$$

applying Lemma 4(ii) with $a = 1/2$ and $b = -1$, we find coefficients

$$\tilde{\alpha}_0 = 1, \quad \tilde{\alpha}_n = 2 \quad (n \geq 1) \qquad\qquad \tilde{\beta}_0 = 2\pi \ , \quad \tilde{\beta}_n = 1 \qquad (n \geq 1) \ .$$

Further, we will define the weight function

$$\hat{w}(x) = \frac{\tilde{w}(x)}{\frac{(r+1)^2}{r} - x} = \frac{1}{\frac{(r+1)^2}{r} - x} \sqrt{\frac{4-x}{x}} \ , \qquad x \in (0, 4) \ .$$

Since $d = (r+1)^2/r > 4$ for $r > 0$, we can apply the case (iv) from Lemma 4. So, we find

$$r_{-1} = \frac{2\pi}{r+1}, \qquad r_n = \frac{1}{r} \quad (n \in \mathbb{N}_0).$$

Hence

$$\hat{\alpha}_0 = \frac{r+1}{r}, \quad \hat{\alpha}_k = 2 \quad (k \in \mathbb{N}), \quad \hat{\beta}_0 = \frac{2\pi}{r+1}, \quad \hat{\beta}_1 = \frac{r+1}{r}, \qquad \hat{\beta}_k = 1 \quad (k \in \mathbb{N}; k \geq 2).$$

Finally, let us denote with $\{S_n(x)\}$ the sequence of monic polynomials orthogonal with respect to the inner product

$$\varphi(f, g) = \int_{\mathbb{R}} f(x)g(x) \ w(x) \ dx \ , \tag{37}$$

where the weight $w(x)$ is defined by

$$w(x) = \frac{r+1}{2\pi r} \ \hat{w}(x) = \frac{r+1}{2\pi r} \cdot \frac{1}{(r+1)^2 - rx} \sqrt{\frac{4-x}{x}} \ , \qquad x \in (0, 4) \ . \tag{38}$$

Applying the case (i) from Lemma 4, we find

$$\alpha_0 = \frac{r+1}{r} \ , \quad \alpha_k = 2 \ (k \in \mathbb{N}), \qquad \beta_0 = \frac{1}{r} \ , \qquad \beta_1 = \frac{r+1}{r} \ , \quad \beta_k = 1 \ (k \in \mathbb{N}; k \geq 2).$$

Their squared norms are

$$\varphi(S_0, \ S_0) = \frac{1}{r} \ , \qquad \varphi(S_n, \ S_n) = \beta_n \beta_{n-1} \ldots \beta_0 = \frac{r+1}{r^2} \qquad (n \in \mathbb{N}) \ . \tag{39}$$

6 The connection with polynomials orthogonal with respect to a discrete Sobolev inner product

Here, we will recall the results from the paper [8] for $\lambda = r/(r-1)$ and $c = \zeta$.

The sequence of monic polynomials $\{Q_n(x)\}$ orthogonal with respect to the inner product

$$\tilde{\varphi}(f, \ g) = \varphi(f, g) \ + \ \frac{1}{\lambda} f(c)g(c) \ , \tag{40}$$

is quite determined by $\{S_n(x)\}$, λ and c.

Lemma 5. *The polynomials* $\{Q_n(x)\}$ *satisfy three-term recurrence relation of the form:*

$$Q_{n+1}(x) = (x - \sigma_n)Q_n(x) - \tau_n Q_{n-1}(x) \quad (n \in \mathbb{N}), \qquad Q_{-1}(x) = 0,\ Q_0(x) = 1. \tag{41}$$

The first few members of the sequence $\{Q_n(x)\}$ are:

$$Q_0(x) = 1, \quad Q_1(x) = x - (r+1), \quad Q_2(x) = x^2 - (r+3)x + (r+1).$$

Hence $\tau_0 = \mu_0 = 1$ and $\tau_1 = r + 1$.

Lemma 6. *The polynomials* $\{S_n(x)\}$ *at the point* ζ *have the following values:*

$$S_0(\zeta) = 1, \qquad S_n(\zeta) = (r+1) \cdot r^{n-1} \quad (n \in \mathbb{N}) \ . \tag{42}$$

Proof. It can be proven by mathematical induction. □

Let us denote by

$$K_m(c,d) = \sum_{j=0}^{m} \frac{S_j(c)S_j(d)}{\varphi(S_j,S_j)}, \qquad \lambda_m = 1 + \frac{K_m(c,c)}{\lambda} \qquad (m \in \mathbb{N}) \ .$$

Here, it is

$$K_m(\zeta,\zeta) = r(r^{2m+1} - 1), \qquad \lambda_m = r^{2m+1} \qquad (m \in \mathbb{N}) \ .$$

Also, in the paper ([8]), it is proven that

$$\tilde{\varphi}(Q_n,Q_n) = \varphi(S_n,S_n)\, \frac{\lambda_n}{\lambda_{n-1}} \qquad (n \in \mathbb{N};\ n \geq 2) \ . \tag{43}$$

Hence

$$\tilde{\varphi}(Q_0,Q_0) = 1, \quad \tilde{\varphi}(Q_1,Q_1) = r+1, \quad \tilde{\varphi}(Q_n,Q_n) = r+1 \qquad (n \in \mathbb{N};\ n \geq 2) \ . \tag{44}$$

Since $\tilde{\varphi}(Q_n,Q_n) = \tau_n\tau_{n-1}\ldots\tau_1\tau_0$, we have

$$\tau_0 = 1, \quad \tau_1 = r+1, \qquad \tau_n = 1 \quad (n \in \mathbb{N};\ n \geq 2) \ .$$

Now, we have all elements for formula (17) and we can compute h_n by

$$h_n = \mu_0^n \tau_1^{n-1} \tau_2^{n-2} \cdots \tau_{n-2}^2 \tau_{n-1} \ .$$

Theorem 7. *The Hankel transform of the sequence* $\{a_n\}$ *defined by (2) is*

$$h_n = (r+1)^{n-1} \qquad (n \in \mathbb{N}) \ .$$

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