A generalization of the concept of q–fractional integrals

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Abstract In this paper, we consider the fractional q–integral with variable lower limit of integration. We prove the semigroup property of these integrals, and a formula of Leibniz type. Finally, we evaluate fractional q–integrals of some functions. The consideration of q–exponential function in that sense leads to q–analogs of Mittag–Leffler function.

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1 Introduction

The fractional calculus is a very suitable tool in describing and solving a lot of problems in sciences, such as physics, electromagnetics, acoustics, viscoelasticity, electrochemistry and material science (see, for example [10]). Of course, for mathematics itself it provides new possibilities such as it is emphasized in [7], [9] and [13]. Their treatment from the point of view of q–calculus can open new perspectives (for example, see [5]).

We begin by recalling a few basic facts [8]. The q–integral is defined by

$$\big(I_{q,0}f\big)(x) = \int_0^x f(t)\, d_qt = x(1-q)\sum_{k=0}^{\infty} f(xq^k)\, q^k \quad (0 \leq |q| < 1)\ , \tag{1}$$

and

$$\big(I_{q,a}f\big)(x) = \int_a^x f(t)\, d_qt = \int_0^x f(t)\, d_qt - \int_0^a f(t)\, d_qt\ . \tag{2}$$

When the lower limit of integration is $a = xq^n$, the q–integral (2) becomes

$$\int_{xq^n}^x f(t)\ d_qt = x(1-q)\sum_{k=0}^{n-1} f(xq^k)q^k\ . \tag{3}$$

We define the iterated q–integral operator $I_{q,a}^n$ by

$$I_{q,a}^0 f = f, \qquad I_{q,a}^n f = I_{q,a}\big(I_{q,a}^{n-1} f\big) \quad (n = 1, 2, 3, \ldots)\ .$$

This can be written in the following form:

$$\big(I_{q,a}^n f\big)(x) = \int_a^x d_qt \int_a^t d_qt_{n-1} \int_a^{t_{n-1}} d_qt_{n-2} \cdots \int_a^{t_2} f(t_1)\ d_qt_1\ .$$

The reduction of this iterated q–integral to a single integral was considered by Al-Salam [3] as a q–analog of Cauchy's formula

$$\big(I_{q,a}^n f\big)(x) = \frac{x^{n-1}}{[n-1]_q!} \int_a^x (qt/x;q)_{n-1}\ f(t)\ d_qt \qquad (n \in \mathbb{N})\ . \tag{4}$$

Al-Salam [2] and Agarwal [1] introduced several types of fractional q–integral operators and fractional q–derivatives, always with the lower limit of integration being 0. Here, we will only mention the following q–analog of the Erdélyi–Kober operator:

$$\big(\mathcal{I}_q^{\eta,\alpha} f\big)(x) = \frac{x^{-(\eta+1)}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\ t^\eta f(t)\ d_qt \qquad (\eta, \alpha \in \mathbb{R}^+)\ .$$

However, in some considerations, such as the construction of a q–Taylor formula or solving of q–differential equation of fractional order, it is of interest to allow that the lower limit of integration is nonzero. Therefore, we define the fractional q–integral by

$$\big(I^{\alpha}_{q,a} f\big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1}\ f(t)\ d_qt \qquad (\alpha \in \mathbb{R}^+)\ .$$

The relationship between these fractional q–integrals is

$$\big(I^{\alpha}_{q,0} f\big)(x) = x^{\alpha}\ \big(\mathcal{I}_q^{0,\alpha} f\big)(x)\ .$$

The permission for the lower limit of integration to take some nonzero value, makes fractional q–calculus even more difficult (see [11]).

In this paper, our purpose is to consider fractional q–integrals with the parametric lower limit of integration. After preliminaries, we present some properties of the q–shifted factorials used in the other sections. In the main parts of the paper, we define the fractional q–integral and the fractional q–derivative and study their properties. In the final section, we derive the fractional q–integrals and q–derivatives of some elementary functions.

2 Preliminaries

In the theory of q–calculus (see [8]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we introduce a q–real number $[a]_q$ and q–shifted factorial by

$$[a]_q := \frac{1-q^a}{1-q}\ , \qquad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i) \qquad \big(a \in \mathbb{R},\ k \in \mathbb{N}\cup\{\infty\}\big)\ .$$

Its natural extension to the reals is

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha \in \mathbb{R})\ . \tag{5}$$

Also, the q–binomial coefficient is given by

$$\begin{bmatrix}\alpha\\ k\end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\ (-1)^k q^{\alpha k} q^{-\binom{k}{2}} \qquad (k \in \mathbb{N},\ \alpha \in \mathbb{R})\ . \tag{6}$$

The following formulas (see, for example, [8] and [4]) will be useful:

$$(\mu;q)_n = \big(q^{1-n}/\mu;q\big)_n\ (-1)^n\ \mu^n\ q^{\binom{n}{2}}\ , \tag{7}$$

$$\frac{(\mu q^{-n};q)_n}{(\nu q^{-n};q)_n} = \frac{(q/\mu;q)_n}{(q/\nu;q)_n}\ \left(\frac{\mu}{\nu}\right)^n\ , \tag{8}$$

$$(\mu;q)_\alpha = \sum_{n=0}^{\infty}(-1)^n \begin{bmatrix}\alpha\\ n\end{bmatrix}_q q^{\binom{n}{2}}\mu^n\ , \tag{9}$$

$$(\mu;q)_{\alpha+n} = (\mu q^\alpha;q)_n\ (\mu;q)_\alpha\ , \tag{10}$$

$$\frac{(\mu q^k;q)_\alpha}{(\mu;q)_\alpha} = \frac{(\mu q^\alpha;q)_k}{(\mu;q)_k}\ , \tag{11}$$

$$(q^{k-n};q)_\alpha = 0 \qquad (n,k \in \mathbb{N},\ n \geq k;\ \mu,\nu,\alpha \in \mathbb{R})\ . \tag{12}$$

The q–gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\ (1-q)^{1-x} \qquad \big(x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}\big)\ , \tag{13}$$

and obviously,

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \qquad \Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x}\ . \tag{14}$$

The q–hypergeometric function [8] is defined as

$${}_2\phi_1\Big(\begin{matrix} a,\ b \\ c \end{matrix}\Big|\ q;x\Big) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n\ (q;q)_n}\ x^n\ .$$

The Heine transformation formula is:

$${}_2\phi_1\Big(\begin{matrix} a,\ b \\ c \end{matrix}\Big|\ q;x\Big) = \frac{(abx/c;q)_\infty}{(x;q)_\infty}\ {}_2\phi_1\Big(\begin{matrix} c/a,\ c/b \\ c \end{matrix}\Big|\ q;abx/c\Big)\ . \tag{15}$$

The q–derivative of a function $f(x)$ is defined by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{x-qx} \quad (x \neq 0)\ , \quad (D_q f)(0) = \lim_{x\to 0}(D_q f)(x)\ ,$$

and the q–derivatives of higher order as follows:

$$D_q^0 f = f\ , \qquad D_q^n f = D_q\big(D_q^{n-1} f\big) \quad (n=1,2,3,\ldots)\ . \tag{16}$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{R}$, we have linearity and product rules, as

$$\begin{aligned} D_q\big(\alpha\ u(x) + \beta\ v(x)\big) &= \alpha\big(D_q u\big)(x) + \beta\big(D_q v\big)(x), \\ D_q\big(u(x)\cdot v(x)\big) &= u(qx)\big(D_q v\big)(x) + v(x)\big(D_q u\big)(x)\ . \end{aligned}$$

In this paper, the q–derivatives of the next functions are very useful examples:

$$D_q\big(x^\lambda (a/x;q)_\lambda\big) = [\lambda]_q x^{\lambda-1}(a/x;q)_{\lambda-1}\ , \tag{17}$$

$$D_q\big(a^\lambda (x/a;q)_\lambda\big) = -[\lambda]_q a^{\lambda-1}(qx/a;q)_{\lambda-1}\ , \tag{18}$$

$$D_q\big(x^\lambda\big) = [\lambda]_q x^{\lambda-1}\ . \tag{19}$$

For the q–integral and q–derivative operators the following relations are valid:

$$\big(D_q^n I_{q,a}^n f\big)(x) = f(x) \quad (n \in \mathbb{N})\ , \tag{20}$$

$$\big(I_{q,a}^n D_q^n f\big)(x) = f(x) - \sum_{k=0}^{n-1} \frac{\big(D_q^k f\big)(a)}{[k]_q!}\ x^k (a/x;q)_k \quad (n \in \mathbb{N})\ . \tag{21}$$

The formula for q–integration by parts is

$$\int_a^b u(x)\big(D_q v\big)(x)\, d_q x = \big[u(x)v(x)\big]_a^b - \int_a^b v(qx)\big(D_q u\big)(x)\, d_q x\ . \tag{22}$$

3 Some useful properties of q–shifted factorials

The following result will be used in proving the semigroup property of the fractional q–integral.

Let us denote

$$S(\alpha,\beta,\mu) = \sum_{n=0}^{\infty} \frac{(\mu q^{1-n};q)_{\alpha-1}\ (q^{1+n};q)_{\beta-1}}{(q;q)_{\alpha-1}\ (q;q)_{\beta-1}}\ q^{\alpha n}\ . \tag{23}$$

Lemma 1 *For $\mu,\alpha,\beta\in\mathbb{R}^+$ the following recurrence relations are valid* [2]*:*

$$\begin{aligned}
(1-q^{\alpha+\beta-1})S(\alpha,\beta,\mu)-(1-\mu q^{\alpha+\beta-1})S(\alpha-1,\beta,\mu) &= 0\\
(1-q^{\alpha+\beta-1})S(\alpha,\beta,\mu)-(1-\mu q^{\alpha+\beta-1})S(\alpha,\beta-1,\mu) &= 0\\
q(1-q^{\alpha+\beta-1})S(\alpha,\beta,\mu)+(1-q)(1-\mu q)D_{q,\mu}S(\alpha,\beta,\mu) &= 0\ .
\end{aligned}$$

Lemma 2 *For $\mu,\alpha,\beta\in\mathbb{R}^+$, the following identity holds:*

$$S(\alpha,\beta,\mu) = \frac{(\mu q;q)_{\alpha+\beta-1}}{(q;q)_{\alpha+\beta-1}}\ . \tag{24}$$

Proof. According to formulas (5) and (8), we have

$$\begin{aligned}
(\mu q^{1-n};q)_{\alpha-1} &= \frac{(\mu q^{1-n};q)_\infty}{(\mu q^{\alpha-n};q)_\infty} = \frac{(\mu q^{1-n};q)_n\ (\mu q;q)_\infty}{(\mu q^{\alpha-n};q)_n\ (\mu q^{\alpha};q)_\infty}\\
&= (\mu q;q)_{\alpha-1}\frac{(\mu^{-1};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n}\ q^{(1-\alpha)n}\ .
\end{aligned}$$

By applying identity (11) to the expression $(q^{1+n};q)_{\beta-1}/(q;q)_{\beta-1}$, we can write $S(\alpha,\beta,\mu)$ in the form

$$\begin{aligned}
S(\alpha,\beta,\mu) &= \frac{(\mu q;q)_{\alpha-1}}{(q;q)_{\alpha-1}}\sum_{n=0}^{\infty}\frac{(q^\beta;q)_n}{(q;q)_n}\frac{(\mu^{-1};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n}\ q^{(1-\alpha)n}q^{\alpha n}\\
&= \frac{(\mu q;q)_{\alpha-1}}{(q;q)_{\alpha-1}}\ {}_2\phi_1\left(\begin{matrix}\mu^{-1},\ q^\beta\\ \mu^{-1}q^{1-\alpha}\end{matrix}\ \Big|\ q;q\right)\ .
\end{aligned}$$

By using (15), we get

$$\begin{aligned}
S(\alpha,\beta,\mu) &= \frac{(\mu q;q)_{\alpha-1}}{(q;q)_{\alpha-1}}\ \frac{(q^{\alpha+\beta};q)_\infty}{(q;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}q^{1-\alpha},\ \mu^{-1}q^{1-\alpha-\beta}\\ \mu^{-1}q^{1-\alpha}\end{matrix}\ \Big|\ q;q^{\alpha+\beta}\right)\\
&= \frac{(\mu q;q)_{\alpha-1}}{(q;q)_{\alpha-1}}\ \frac{1}{(q;q)_{\alpha+\beta-1}}\ \sum_{n=0}^{\infty}\frac{(q^{1-\alpha};q)_n\ (\mu^{-1}q^{1-\alpha-\beta};q)_n}{(q;q)_n\ (\mu^{-1}q^{1-\alpha};q)_n}\ q^{(\alpha+\beta)n}\ .
\end{aligned}$$

[2] For the properties exposed in this lemma, we are thankful to W. Koepf who observed them by his Maple package `qsum` [6].

According to (7), the following is valid:

$$\frac{(\mu^{-1}q^{1-\alpha-\beta};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n} = \frac{(\mu q^{\alpha+\beta-n};q)_n}{(\mu q^{\alpha-n};q)_n}\ q^{-\beta n} = \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\ q^{-\beta n}$$
$$= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\ q^{-\beta n}$$
$$= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ (\mu q^{\alpha+\beta-n};q)_{-\beta}\ q^{-\beta n}\ .$$

Hence

$$S(\alpha,\beta,\mu) = \frac{(\mu q;q)_{\alpha+\beta-1}}{(q;q)_{\alpha-1}\ (q;q)_{\alpha+\beta-1}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\ q^{\alpha n} (\mu q^{\alpha+\beta-n};q)_{-\beta}\ .$$

If we use formulas (6) and (9), the previous sum becomes

$$\sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\ q^{\alpha n}(\mu q^{\alpha+\beta-n};q)_{-\beta}$$
$$= \sum_{n=0}^{\infty} \begin{bmatrix}\alpha-1\\ n\end{bmatrix}_q (-1)^n\ q^{-(\alpha-1)n} q^{\binom{n}{2}}\ q^{\alpha n} \sum_{k=0}^{\infty}(-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta-n}\right)^k$$
$$= \sum_{k=0}^{\infty}(-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \sum_{n=0}^{\infty}(-1)^n \begin{bmatrix}\alpha-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}\ \left(q^{1-k}\right)^n$$
$$= \sum_{k=0}^{\infty}(-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \left(q^{1-k};q\right)_{\alpha-1} = (q;q)_{\alpha-1}\ .$$

This relation is valid since $\left(q^{1-k};q\right)_{\alpha-1} = 0$ for $k = 1,2,\ldots$. Finally, the following identity holds:

$$S(\alpha,\beta,\mu) = \frac{(\mu q;q)_{\alpha+\beta-1}}{(q;q)_{\alpha-1}\ (q;q)_{\alpha+\beta-1}}\ (q;q)_{\alpha-1} = \frac{(\mu q;q)_{\alpha+\beta-1}}{(q;q)_{\alpha+\beta-1}}\ .\ \square$$

4 The fractional q–integral

In all further considerations we assume that the functions are defined in an interval $(0,b)$ $(b>0)$, and $a \in (0,b)$ is an arbitrary fixed point. Also, we presume that the required q–derivatives and q–integrals exist and that the series, mentioned in the proofs, converge.

The next definition gives a generalization of the formula (4).

Definition 1 The *fractional q–integral* is

$$\left(I_{q,a}^{\alpha} f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1}\ f(t)\ d_q t \qquad (\alpha \in \mathbb{R}^+)\ . \tag{25}$$

Since

$$\lim_{q\nearrow 1} x^{\alpha-1}(qt/x;q)_{\alpha-1} = (x-t)^{\alpha-1},$$

the fractional integral (see, for example [10]) occurs as limit case of (25) when $q \nearrow 1$.

By using formula (9), the integral (25) can be written as

$$\big(I_{q,a}^{\alpha} f\big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix}\alpha-1\\ k\end{bmatrix}_q q^{\binom{k+1}{2}} x^{-k} \int_a^x t^k f(t)\ d_q t \quad (\alpha \in \mathbb{R}^+)\ .$$

Putting $\alpha = 1$ in the previous relation, we get the q–integral (2).

Lemma 3 *For $\alpha \in \mathbb{R}^+$, the following is valid:*

$$\big(I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+1} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} x^{\alpha}(a/x;q)_{\alpha} \qquad (0 < a < x)\ .$$

Proof. According to formula (18), the q–derivative over the variable t is

$$D_q\big(x^{\alpha}(t/x;q)_{\alpha}\big) = -[\alpha]_q x^{\alpha-1}(qt/x;q)_{\alpha-1}\ .$$

Using the q–integration by parts (22), we obtain

$$\begin{aligned}\big(I_{q,a}^{\alpha} f\big)(x) &= -\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_a^x D_q\big(x^{\alpha}(t/x;q)_{\alpha}\big) f(t) d_q t \\ &= \frac{1}{\Gamma_q(\alpha+1)} \Big(x^{\alpha}(a/x;q)_{\alpha} f(a) + \int_a^x x^{\alpha}(qt/x;q)_{\alpha}\big(D_q f\big)(t) d_q t\Big) \\ &= \big(I_{q,a}^{\alpha+1} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} x^{\alpha}(a/x;q)_{\alpha}\ . \ \square\end{aligned}$$

Lemma 4 *For $\alpha, \beta \in \mathbb{R}^+$, the following is valid:*

$$\int_0^a (qt/x;q)_{\beta-1}\big(I_{q,a}^{\alpha} f\big)(t) d_q t = 0 \qquad (0 < a < x)\ .$$

Proof. Using formulas (3) and (12), for $n \in \mathbb{N}_0$, we have

$$\begin{aligned}\big(I_{q,a}^{\alpha} f\big)(aq^n) &= \frac{1}{\Gamma_q(\alpha)} \int_a^{aq^n} (aq^n)^{\alpha-1}\big((qu)/(aq^n);q\big)_{\alpha-1} f(u) d_q u \\ &= \frac{-a^{\alpha}(1-q)}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} (q^n)^{\alpha-1}(q^{j+1-n};q)_{\alpha-1} f(aq^j) q^j = 0\ .\end{aligned}$$

On the other hand, according to the definition of q–integral, we have

$$\int_0^a (qt/x;q)_{\beta-1}\big(I_{q,a}^{\alpha} f\big)(t) d_q t = a(1-q) \sum_{n=0}^{\infty} (aq^{n+1}/x;q)_{\beta-1}\big(I_{q,a}^{\alpha} f\big)(aq^n) q^n\ ,$$

which is equal to zero. $\square$

Theorem 5 *Let $\alpha, \beta \in \mathbb{R}^+$. The q–fractional integration has the following semigroup property:*

$$\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha+\beta} f\right)(x) \qquad (0 < a < x) \ .$$

Proof. By previous lemma, we have

$$\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \frac{x^{\beta-1}}{\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta-1} \left(I_{q,a}^{\alpha} f\right)(t) d_q t,$$

i.e.,

$$\begin{aligned}\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) &= \frac{x^{\beta-1}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta-1} \ t^{\alpha-1} \int_0^t (qu/t;q)_{\alpha-1} \ f(u) d_q u \\ &- \frac{x^{\beta-1}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta-1} \ t^{\alpha-1} \int_0^a (qu/t;q)_{\alpha-1} \ f(u) d_q u \ .\end{aligned}$$

Due to equality

$$\left(I_{q,0}^{\beta} I_{q,0}^{\alpha} f\right)(x) = \left(I_{q,0}^{\alpha+\beta} f\right)(x) \ ,$$

proved in [1], we conclude that

$$\begin{aligned}\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) &= \left(I_{q,0}^{\alpha+\beta} f\right)(x) \\ &- \frac{x^{\beta-1}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta-1} \ t^{\alpha-1} \int_0^a (qu/t;q)_{\alpha-1} \ f(u) d_q u \ .\end{aligned}$$

Furthermore, we can write

$$\begin{aligned}\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) &= \left(I_{q,a}^{\alpha+\beta} f\right)(x) + \frac{x^{\alpha+\beta-1}}{\Gamma_q(\alpha+\beta)} \int_0^a (qt/x;q)_{\alpha+\beta-1} f(t) d_q t \\ &- \frac{x^{\beta-1}}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta-1} t^{\alpha-1} \int_0^a (qu/t;q)_{\alpha-1} f(u) d_q u,\end{aligned}$$

whence

$$\left(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\right)(x) = \left(I_{q,a}^{\alpha+\beta} f\right)(x) + a(1-q) \sum_{j=0}^{\infty} c_j f(aq^j) q^j,$$

with

$$\begin{aligned}c_j &= \frac{x^{\alpha+\beta-1}(aq^{j+1}/x;q)_{\alpha+\beta-1}}{\Gamma_q(\alpha+\beta)} \\ &- \frac{x^{\alpha+\beta-1}(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{n=0}^{\infty} (q^{n+1};q)_{\beta-1} \ q^{n(\alpha-1)} (aq^{j+1-n}/x;q)_{\alpha-1} \ q^n.\end{aligned}$$

Using the formulas (11), (12) and (14), we get

$$\begin{aligned}c_j &= \left((1-q)x\right)^{\alpha+\beta-1} \\ &\times \left\{ \frac{(aq^{j+1}/x;q)_{\alpha+\beta-1}}{(q;q)_{\alpha+\beta-1}} - \sum_{n=0}^{\infty} \frac{(q^{n+1};q)_{\beta-1}}{(q;q)_{\beta-1}} \ \frac{(aq^{j+1-n}/x;q)_{\alpha-1}}{(q;q)_{\alpha-1}} \ q^{n\alpha} \right\} \ .\end{aligned}$$

By substituting $\mu = q^j a/x$ in (24), we see that $c_j = 0$ for all $j \in \mathbb{N}$, which completes the proof. □

5 Leibniz–type formula for fractional q–integrals

The q–analog of fractional Leibniz formula for q–integrals

$$I_{q,0}^{\alpha}\big(f(x)\ g(x)\big) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q \big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big)\big(I_{q,0}^{\alpha+m} g\big)(x) \tag{26}$$

was proven by W.A. Al-Salam and A. Verma [4]. Notice that it contains only the case $a = 0$. Our purpose is to formulate and prove it for arbitrary $a \in \mathbb{R}^+$.

Theorem 6 *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < b$, the fractional q–Leibniz formula is*

$$I_{q,a}^{\alpha}\big(f(x)\ g(x)\big) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q \big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big)\big(I_{q,a}^{\alpha+m} g\big)(x)\ . \tag{27}$$

Proof. By definition of fractional q–integral, we can write

$$\begin{aligned} I_{q,a}^{\alpha}\big(f(x)\ g(x)\big) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} f(t)g(t)\ d_qt \\ &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \Big(\int_0^x (qt/x;q)_{\alpha-1} f(t)g(t)\ d_qt - \int_0^a (qt/x;q)_{\alpha-1} f(t)g(t)\ d_qt \Big) \\ &= I_{q,0}^{\alpha}\big(f(x)\ g(x)\big) - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^a (qt/x;q)_{\alpha-1} f(t)g(t)\ d_qt\ . \end{aligned}$$

Since

$$\big(I_{q,a}^{\alpha+m} g\big)(x) = \big(I_{q,0}^{\alpha+m} g\big)(x) - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^a (qt/x;q)_{\alpha-1}\ g(t)\ d_qt,$$

and by (26), we have

$$I_{q,a}^{\alpha}\big(f(x)\ g(x)\big) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q \big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big)\big(I_{q,a}^{\alpha+m} g\big)(x) - \Theta(x),$$

where

$$\begin{aligned} \Theta(x) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q \big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big) \frac{x^{\alpha+m-1}}{\Gamma_q(\alpha+m)} \int_0^a (qt/x;q)_{\alpha+m-1} g(t)\ d_qt \\ - \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^a (qt/x;q)_{\alpha-1} f(t)g(t)\ d_qt\ . \end{aligned}$$

We can write it in the integral form

$$\Theta(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^a (qt/x;q)_{\alpha-1} g(t)\Psi(t)\ d_qt\ ,$$

where

$$\Psi(t)=\sum_{m=0}^{\infty}\frac{(-1)^m q^{-\binom{m}{2}-m\alpha}}{[m]_q!}\big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big)\frac{x^m\Gamma_q(\alpha)(qt/x;q)_{\alpha+m-1}}{\Gamma_q(\alpha+m)(qt/x;q)_{\alpha-1}}-f(t).$$

Using the properties (6) and (10), we have

$$\Psi(t)=\sum_{m=0}^{\infty}(-1)^m\frac{q^{-\binom{m}{2}-m\alpha}}{[m]_q!}\big(D_q^m f\big)\big(xq^{-(\alpha+m)}\big)x^m(q^{\alpha}t/x;q)_m-f(t).$$

This infinite sum is modified q–Taylor expansion (see [4])

$$f(t)=\sum_{m=0}^{\infty}(-1)^m\ \frac{q^{-\binom{m}{2}-m\alpha}}{[m]_q!}\big(D_q^m f\big)\big(zq^{-m}\big)\ z^m(t/z;q)_m \tag{28}$$

of the function $f(t)$ at the point $z=xq^{-\alpha}$. Hence we conclude that $\Psi(t)\equiv 0$ wherefrom $\Theta(t)\equiv 0$. $\Box$

6 The fractional q–integrals of some functions

We use the previous results to evaluate the fractional q–integrals of some well-known functions in the explicit form.

Corollary 7 *If $\alpha\in\mathbb{R}^+$, $\lambda\in(-1,\infty)$, then:*

$$I_{q,a}^{\alpha}\big(x^{\lambda}(a/x;q)_{\lambda}\big)=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}\ x^{\alpha+\lambda}(a/x;q)_{\alpha+\lambda}\qquad(0<a<x)\ . \tag{29}$$

Proof. For $\lambda\neq 0$, according to the definition (25), we have

$$\begin{aligned}&I_{q,a}^{\alpha}\big(x^{\lambda}(a/x;q)_{\lambda}\big)\\&\qquad=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\Big(\int_0^x(qt/x;q)_{\alpha-1}t^{\lambda}(a/t;q)_{\lambda}d_qt-\int_0^a(qt/x;q)_{\alpha-1}t^{\lambda}(a/t;q)_{\lambda}d_qt\Big).\end{aligned}$$

Also, the following is valid:

$$\int_0^a(qt/x;q)_{\alpha-1}\ t^{\lambda}(a/t;q)_{\lambda}d_qt=a^{\lambda+1}(1-q)\sum_{k=0}^{\infty}(aq^{k+1}/x;q)_{\alpha-1}\ q^{k\lambda}(q^{-k};q)_{\lambda}\ q^k\ .$$

It vanishes because of (12). Hence, according to definition (1), we get

$$\begin{aligned}&\int_0^x(qt/x;q)_{\alpha-1}\ t^{\lambda}(a/t;q)_{\lambda}\ d_qt\\&\qquad\qquad=x^{\lambda+1}(1-q)\sum_{k=0}^{\infty}(q^{1+k};q)_{\alpha-1}\ \big(a/(xq^k);q\big)_{\lambda}\ q^{(\lambda+1)k}\ .\end{aligned}$$

In view of (23), the previous formula gets the form

$$\int_0^x (qt/x;q)_{\alpha-1}\ t^\lambda (a/t;q)_\lambda\ d_qt$$
$$= (1-q)\ x^{\lambda+1}\ (q;q)_{\alpha-1}\ (q;q)_\lambda\ S\Big(\lambda+1,\alpha,a/(qx)\Big)\ .$$

By using (24), we get

$$\int_0^x (qt/x;q)_{\alpha-1} t^\lambda (a/t;q)_\lambda d_qt = (1-q)\frac{(q;q)_{\alpha-1}(q;q)_\lambda}{(q;q)_{\alpha+\lambda}}\ x^{\lambda+1}(a/x;q)_{\alpha+\lambda},$$

and applying (14), we obtain the required formula for $I^\alpha_{q,a}\big(x^\lambda(a/x;q)_\lambda\big)$ when $\lambda \neq 0$.

In case when $\lambda = 0$, using q–integration by parts (22), we have

$$(I^\alpha_{q,a}\mathbf{1})(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1} d_qt = \frac{1}{\Gamma_q(\alpha)}\int_a^x \frac{D_q\big(x^\alpha(t/x;q)_\alpha\big)}{-[\alpha]_q} d_qt$$
$$= \frac{-1}{\Gamma_q(\alpha+1)}\int_a^x D_q\big(x^\alpha(t/x;q)_\alpha\big) d_qt = \frac{1}{\Gamma_q(\alpha+1)} x^\alpha(a/x;q)_\alpha\ .\ \square$$

Corollary 8 *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$, $n \in \mathbb{N}_0$, and $0 < a < x$, the following is valid:*

$$I^\alpha_{q,a}\big(x^\lambda\big) = \frac{a^\lambda}{\Gamma_q(\alpha+1)}\ x^\alpha(a/x;q)_\alpha + [\lambda]_q I^{\alpha+1}_{q,a}\big(x^{\lambda-1}\big)\ ,$$

$$I^\alpha_{q,a}\big(x^n\big) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!\ a^{n-k}}{\Gamma_q(\alpha+k+1)}\ x^{\alpha+k}(a/x;q)_{\alpha+k}\ . \tag{30}$$

Proof. The first relation follows from the definition of fractional integral and the formula for q–integration by parts (22). Especially, if $\lambda = n \in \mathbb{N}_0$, by repeated n times use of previous formula, we get

$$I^\alpha_{q,a}\big(x^n\big) = \frac{a^n}{\Gamma_q(\alpha+1)}\ x^\alpha(a/x;q)_\alpha + [n]_q\frac{a^{n-1}}{\Gamma_q(\alpha+2)}\ x^{\alpha+1}(a/x;q)_{\alpha+1} + \dots$$
$$+ [n]_q\cdots[2]_q\frac{a^{n-1}}{\Gamma_q(\alpha+n)}\ x^{\alpha+n-1}(a/x;q)_{\alpha+n-1} + [n]_q!\big(I^{\alpha+n}_{q,a}\mathbf{1}\big)(x)\ .$$

Using formula (29) for $\lambda = 0$, i.e,

$$\big(I^{\alpha+n}_{q,a}\mathbf{1}\big)(x) = \frac{1}{\Gamma_q(\alpha+n+1)} x^{\alpha+n}(a/x;q)_{\alpha+n}\ ,$$

we complete the proof of equality (30). $\square$

The formula (30) can be written in the equivalent form

$$I^\alpha_{q,a}\big(x^n\big) = (1-q)^\alpha \sum_{k=0}^n a^{n-k}\ (q^{n-k+1};q)_k\ \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}}\ . \tag{31}$$

In q–calculus (see [8]) the following functions are well–known as analogues of the exponential function:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}\, x^n \qquad (|x|<1)\ , \tag{32}$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n}\, x^n = (-x;q)_\infty \qquad (x\in\mathbb{R})\ . \tag{33}$$

Corollary 9 *For $\alpha\in\mathbb{R}^+$ and $0<a<x<1$, the following q–integral is valid:*

$$I^{\alpha}_{q,a}\big(e_q(x)\big) = (1-q)^{\alpha}\ e_q(a)\sum_{n=0}^{\infty}\frac{x^{\alpha+n}(a/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}}\ .$$

Proof. According to definition (32) and formula (30), we have

$$I^{\alpha}_{q,a}\big(e_q(x)\big) = \sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{[k]_q!\ a^{n-k}}{\Gamma_q(\alpha+k+1)}\ x^{\alpha+k}(a/x;q)_{\alpha+k}\ .$$

By appropriate transformation of the sum, it becomes

$$\begin{aligned} I^{\alpha}_{q,a}\big(e_q(x)\big) &= \sum_{n=0}^{\infty}\sum_{i=n}^{\infty}\frac{a^{i-n}}{(q;q)_{i-n}}\ \frac{1}{(1-q)^n\Gamma_q(\alpha+n+1)}\ x^{\alpha+n}(a/x;q)_{\alpha+n} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{a^{j}}{(q;q)_{j}}\right)\frac{1}{(1-q)^n\Gamma_q(\alpha+n+1)}\ x^{\alpha+n}(a/x;q)_{\alpha+n} \\ &= e_q(a)\sum_{n=0}^{\infty}\frac{1}{(1-q)^n\Gamma_q(\alpha+n+1)}\ x^{\alpha+n}(a/x;q)_{\alpha+n}\ . \end{aligned}$$

In view of (13) and (5), we can write

$$(1-q)^n\Gamma_q(\alpha+n+1) = (1-q)^{-\alpha}(q;q)_{\alpha+n}\ ,$$

which completes the proof. □

Corollary 10 *For $\alpha\in\mathbb{R}^+$ and $0<a<x$, the following formula holds:*

$$I^{\alpha}_{q,a}\big(E_q(x)\big) = (1-q)^{\alpha}\ E_q(a)\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(-a;q)_n}\ \frac{x^{\alpha+n}(a/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}}\ .$$

Proof. Using definition (33), as in the proof of the previous lemma, we get

$$\begin{aligned} I^{\alpha}_{q,a}\big(E_q(x)\big) &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{q^{\binom{j+n}{2}}a^{j}}{(q;q)_{j}}\right)\frac{1}{(1-q)^n\Gamma_q(\alpha+n+1)}\ x^{\alpha+n}(a/x;q)_{\alpha+n} \\ &= (1-q)^{\alpha}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{q^{\binom{j}{2}}(aq^n)^{j}}{(q;q)_{j}}\right)\frac{q^{\binom{n}{2}}}{(q;q)_{\alpha+n}}\ x^{\alpha+n}(a/x;q)_{\alpha+n}\ . \end{aligned}$$

Having in mind that

$$\sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(aq^n)^j}{(q;q)_j} = E_q(aq^n) = (-aq^n;q)_\infty = \frac{(-a;q)_\infty}{(-a;q)_n} = \frac{E_q(a)}{(-a;q)_n} \ ,$$

the statement is proven. □

The two previous corollaries give the hint to define (see [12]) the special functions which are q–analogs of Mittag–Lefler function

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+\beta)} \qquad (\beta \in \mathbb{C};\ \mathrm{Re}(\beta) > 0) \ .$$

We shall call the function

$$e_{\beta,q}(x;c) = \sum_{n=0}^{\infty} \frac{x^{n+\beta-1}(c/x;q)_{n+\beta-1}}{(q;q)_{n+\beta-1}} \qquad (|c| < |x|) \ ,$$

$$\big(q,x,c,\beta \in \mathbb{C};\ \mathrm{Re}(\beta) > 0, \quad |q| < 1\big) \tag{34}$$

the *small q–Mittag–Lefler function.*

Similarly, we define the *big q–Mittag–Lefler function* by

$$E_{\beta,q}(x;c) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^{n+\beta-1}(c/x;q)_{n+\beta-1}}{(-c;q)_n \ (q;q)_{n+\beta-1}} \ ,$$

under the same conditions (34).

In the limit case, we get

$$\lim_{c\to 0}\lim_{q\to 1} e_{\beta,q}\big((1-q)x;c\big) = \lim_{c\to 0}\lim_{q\to 1} E_{\beta,q}\big((1-q)x;c\big) = x^{\beta-1}E_\beta(x) \ .$$

Especially,

$$e_{1,q}(x;0) = e_q(x), \qquad E_{1,q}(x;0) = E_q(x) \ .$$

Now, we can write the conclusions of Corollary 9 and Corollary 10 in the form:

$$I^\alpha_{q,a}\big(e_q(x)\big) = (1-q)^\alpha \ e_q(a) \ e_{q,\alpha+1}(x;a)$$
$$I^\alpha_{q,a}\big(E_q(x)\big) = (1-q)^\alpha \ E_q(a) \ E_{q,\alpha+1}(x;a) \ ,$$

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