ON SEMILATTICES OF ARCHIMEDEAN SEMIGROUPS
- A SURVEY *

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We provide an exposition of the approach to the structure theory of semigroups through semilattice decompositions. This approach was introduced by A. H. Clifford, extended by T. Tamura and others including M. Petrich, and, in particular, by the members of the Niš school semigroup theory, headed by S. Bogdanović. An emphasis is placed on the author’s results some published in various articles, and some appearing for the first time in print in her book "Semilattices of Archimedean Semigroups".

Introduction

Among methods used for describing a structure of a certain type of semigroup there is a method of decomposition of a semigroup, based on the partition of a semigroup, describing each of the components’ structure and establishing connections between them. In fact, semilattice decompositions of semigroups play the central role in this paper.

A congruence \( \xi \) of a semigroup \( S \) is a semilattice congruence of \( S \) if the factor \( S/\xi \) is a semilattice. The partition and the factor determined by such \( \xi \) are called semilattice decomposition and semilattice homomorphic image of \( S \), respectively. A semigroup \( S \) is semilattice indecomposable if \( S \times S \) is semilattice indecomposable the only semilattice congruence of \( S \). Let us denote a \( \xi \)-class of \( S \) by \( S_i, i \in I \), where \( I \) is a semilattice isomorphic to \( S/\xi \), with isomorphism defined by \( i \mapsto S_i \). If \( K \) is a certain class of semigroups, and if \( S_i \in K \), for every \( i \in I \), then we say that \( S \) is a semilattice \( I \) of semigroups \( S_i, i \in I \), from a class \( K \).

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Semi lattice decompositions of semigroups was introduced by A. H. Clifford in 1941 (see [25]). Since that time several authors have been working on the topic. Special contributions to the development of the theory of semi-lattice decompositions were made by T. Tamura, and co-authors N. Kimura and J. Shafer. The series of their papers began in 1954, [84], where semi-lattice decompositions of commutative semigroups were considered. The existence of the greatest semilattice decomposition of an arbitrary semigroup was established in 1955 (see [85,89]). The fundamental result that components in the greatest semilattice decomposition of a semigroup are semi-lattice-indecomposable, was proved in 1956 (see [79]). Various characterizations of the smallest semilattice congruence on a semigroup can be found also in [80,82]. Among the authors who also gave contributions to the theory of semilattice decompositions are also M. Petrich (see [61,62]), M. S. Putcha (see [70]), R. Šulka (see [78]), M. Ćirić and S. Bogdanović (see [28,29,30]).

Here, we are interested, in particular, in the decomposability of a certain type of semigroups into a semilattice of archimedean semigroups. The most of results on decompositions of completely \(\pi\)-regular semigroups into semilattices of archimedean semigroups obtained till 1992 were cumulated in [12]. The present survay is however not a mere continuation of [12]. It is structured as follows: Section 1 gives an overview of results concerning different types of regularity, archimedean semigroups and semilattices of archimedean semigroups. Beside of the fact that one can find a few overlaps with [12], this section is self-contained. Sections 2-6 are based on author’s original results some appearing in [55], some published in various articles ([19,20,57]), some appearing for the first time in print in [56].

The main characterization of the approach given in Sections 2-6 is making connections of semilattice decomposition of a certain type of semigroup into archimedean components, on the one hand, and equalities between (generalized) regular subsets and Green’s subsets of such semigroup on the other one. Following that point of view, we have the new characterizations of semilattices of nil-extensions of simple semigroups (Theorem 2.5) in Section 2, which contains some basic results to be used in the other parts of this paper. In Section 3, which is the core of the survay, we consider left quasi-\(\pi\)-regular semigroups (Theorem 3.1), and give conditions under which we can decompose such semigroups (Theorem 3.11). The semilattices of left strongly archimedean semigroups, as we call left quasi-\(\pi\)-regular semigroups which are also semilattices of archimedean semigroups, are generalizations of some already known semigroups such as: semilattices
of left completely archimedean semigroups (Section 4), semilattices of nil-extensions of simple and regular semigroups (Section 5) and semilattices of completely archimedean semigroups (Section 6). The proofs of results given in these sections can be found in [56], so that is why they are omitted here.

Throughout this paper \( \mathbb{N} \) will denote the set of positive integers. Let \( S \) be a semigroup. If \( S \) contains an idempotent, then we will denote by \( E(S) \) a set of its idempotents. The intersection of all subsemigroups of \( S \) containing \( E(S) \) is the \textit{idempotent-generated} subsemigroup of \( S \), denoted by \( \langle E(S) \rangle \).

If \( a \in S \), then non-empty intersections of ideals (left ideals, right ideals) of \( S \) which contain \( a \) is \textit{principal ideal} (principal left ideal, principal right ideal) of \( S \) generated by \( a \) and denoted by \( J(a) \) \( (L(a), R(a)) \). By using principal ideals of certain elements of a semigroup \( S \), we can define various very important relations on \( S \). Let \( a, b \in S \). Division relations on \( S \) are defined by: \( a | b \iff b \in J(a), a \mid b \iff b \in L(a), a \mid b \iff b \in R(a), a \mid b \iff a \mid b, \& a \mid b \). Relations \( J, L, R, D \) and \( H \) defined on \( S \) by \( a J b \iff J(a) = J(b), a L b \iff L(a) = L(b), a R b \iff R(a) = R(b), D = LR = RL, H = L \cap R \) are well-known Green’s relations or Green’s equivalences. For any element \( a \) of a semigroup \( S \) and \( T \in \{ J, L, R, D, H \} \), the \( T \)-class of \( S \) containing \( a \) is denoted by \( T_a \). Now, once again, we use the division relations for defining the following relations on \( S \): \( a \rightarrow b \iff (\exists n \in \mathbb{N}) a | b^n, a \rightarrow l b \iff (\exists n \in \mathbb{N}) a \mid l b^n, a \rightarrow r b \iff (\exists n \in \mathbb{N}) a \mid r b^n, a \rightarrow t b \iff a \rightarrow l b \& a \rightarrow r b \).

Given an arbitrary relation \( \rho \) on \( S \), then \( a \tau(\rho) b \iff (\exists n \in \mathbb{N}) a^n \rho b^n \) is a relation on \( S \) called the \( \tau \)-\textit{radical} of \( \rho \).

Before we proceed we will present tables and diagrams that visualize connections of semigroups which will be studied together with their semilattices in Sections 1 - 6.

<table>
<thead>
<tr>
<th>NOTATION</th>
<th>CLASS OF SEMIGROUPS</th>
<th>CHARACTERIZATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>semisimple</td>
<td>( a \in S_a S_a )</td>
</tr>
<tr>
<td>LQR</td>
<td>left quasi-regular</td>
<td>( a \in S_a )</td>
</tr>
<tr>
<td>RQR</td>
<td>right quasi-regular</td>
<td>( a \in S_a S )</td>
</tr>
<tr>
<td>CQR</td>
<td>completely quasi-regular</td>
<td>( a \in S_a S_a )</td>
</tr>
<tr>
<td>IRR</td>
<td>intra-regular</td>
<td>( a \in S_a )</td>
</tr>
<tr>
<td>LRR</td>
<td>left regular</td>
<td>( a \in S_a )</td>
</tr>
<tr>
<td>RRR</td>
<td>right regular</td>
<td>( a \in S_a )</td>
</tr>
<tr>
<td>RR</td>
<td>regular</td>
<td>( a \in S_a )</td>
</tr>
<tr>
<td>CR</td>
<td>completely regular</td>
<td>( a \in S_a )</td>
</tr>
</tbody>
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### Table 2. (Generalized) $\pi$-regular semigroups

<table>
<thead>
<tr>
<th>NOTATION</th>
<th>CLASS OF SEMIGROUPS</th>
<th>CHARACTERIZATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi SS$</td>
<td>$\pi$-semisimple</td>
<td>$a^n \in S a^n S$</td>
</tr>
<tr>
<td>$LQ\pi R$</td>
<td>left quasi-$\pi$-regular</td>
<td>$a^n \in S a^n S$</td>
</tr>
<tr>
<td>$RQ\pi R$</td>
<td>right quasi-$\pi$-regular</td>
<td>$a^n \in a^n S a^n S$</td>
</tr>
<tr>
<td>$CQ\pi R$</td>
<td>completely quasi-$\pi$-regular</td>
<td>$a^n \in a^n S a^n S \cap S a^n S a^n$</td>
</tr>
<tr>
<td>$I\pi R$</td>
<td>intra-$\pi$-regular</td>
<td>$a^n \in S a^n S$</td>
</tr>
<tr>
<td>$L\pi R$</td>
<td>left $\pi$-regular</td>
<td>$a^n \in S a^n S$</td>
</tr>
<tr>
<td>$R\pi R$</td>
<td>right $\pi$-regular</td>
<td>$a^n \in a^n S a^n S$</td>
</tr>
<tr>
<td>$\pi R$</td>
<td>$\pi$-regular</td>
<td>$a^n \in a^n S a^n S$</td>
</tr>
<tr>
<td>$C\pi R = UG$</td>
<td>completely $\pi$-regular</td>
<td>$a^n \in a^n S a^n S$</td>
</tr>
</tbody>
</table>

### Figure 1.

Table 3. Simple semigroups

<table>
<thead>
<tr>
<th>NOTATION</th>
<th>CLASS OF SEMIGROUPS</th>
<th>CHARACTERIZATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>simple</td>
<td>$a \in S b S$</td>
</tr>
<tr>
<td>$LS$</td>
<td>left simple</td>
<td>$a \in S b S$</td>
</tr>
<tr>
<td>$RS$</td>
<td>right simple</td>
<td>$a \in b S$</td>
</tr>
<tr>
<td>$G$</td>
<td>group</td>
<td>$a \in b S \cap S b S$</td>
</tr>
<tr>
<td>$LSS$</td>
<td>left strongly simple</td>
<td>$a \in S b S a$</td>
</tr>
<tr>
<td>$RSS$</td>
<td>right strongly simple</td>
<td>$a \in a S b S$</td>
</tr>
<tr>
<td>$SSS$</td>
<td>strongly simple</td>
<td>$a \in a S b S \cap S b S a$</td>
</tr>
<tr>
<td>$LCS$</td>
<td>left completely simple</td>
<td>$a \in S b a$</td>
</tr>
<tr>
<td>$RCS$</td>
<td>right completely simple</td>
<td>$a \in a b S$</td>
</tr>
<tr>
<td>$SR$</td>
<td>simple and regular</td>
<td>$a \in a S b S a$</td>
</tr>
<tr>
<td>$CS$</td>
<td>completely simple</td>
<td>$a \in a b S a$</td>
</tr>
</tbody>
</table>
Table 4. Archimedean semigroups

<table>
<thead>
<tr>
<th>NOTATION</th>
<th>CLASS OF SEMIGROUPS</th>
<th>CHARACTERIZATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}$</td>
<td>archimedean</td>
<td>$a^n \in SbS$</td>
</tr>
<tr>
<td>$\mathcal{L}A$</td>
<td>left archimedean</td>
<td>$a^n \in Sb$</td>
</tr>
<tr>
<td>$\mathcal{R}A$</td>
<td>right archimedean</td>
<td>$a^n \in bS$</td>
</tr>
<tr>
<td>$T_\mathcal{A}$</td>
<td>$t$-archimedean</td>
<td>$a^n \in bS \cap Sb$</td>
</tr>
<tr>
<td>$NES$</td>
<td>nil-extension of simple</td>
<td>$a^n \in Sb^n S$</td>
</tr>
<tr>
<td>$N_{LS}$</td>
<td>nil-extension of left simple</td>
<td>$a^n \in Sb^n$</td>
</tr>
<tr>
<td>$N_{ERS}$</td>
<td>nil-extension of right simple</td>
<td>$a^n \in b^n S$</td>
</tr>
<tr>
<td>$NEG$</td>
<td>nil-extension of group</td>
<td>$a^n \in b^n Sb^n$</td>
</tr>
<tr>
<td>$LSA$</td>
<td>left strongly archimedean</td>
<td>$a^n \in Sbh^n$</td>
</tr>
<tr>
<td>$RSA$</td>
<td>right strongly archimedean</td>
<td>$a^n \in SbSa^n$</td>
</tr>
<tr>
<td>$SA$</td>
<td>strongly archimedean</td>
<td>$a^n \in a^n SbSa^n$</td>
</tr>
<tr>
<td>$L\mathcal{C}A$</td>
<td>left completely archimedean</td>
<td>$a^n \in Sb^n \cap SbSa^n$</td>
</tr>
<tr>
<td>$R\mathcal{C}A$</td>
<td>right completely archimedean</td>
<td>$a^n \in a^n bSb^n$</td>
</tr>
<tr>
<td>$\pi A$</td>
<td>$\pi$-regular and archimedean</td>
<td>$a^n \in a^n SbSa^n$</td>
</tr>
<tr>
<td>$\mathcal{C}A$</td>
<td>completely archimedean</td>
<td>$a^n \in a^n bSa^n$</td>
</tr>
</tbody>
</table>

According to the pictures we have to emphasize here that the ascending lines go from a class to a larger class, or equivalently, from a condition to a weaker condition. Thus, we have an ordered set, which might look like a lattice, but, it is not so: for example, $\mathcal{A}$ is not the union of $\mathcal{L}A$ and $\mathcal{R}A$, or $G$ is not the intersection of $LS$ and $RSS$. (This does not in contradict to the fact that sometimes a class, which looks like the intersection of two larger classes on the diagram, is indeed equal to their intersection.)
All the classes in the first diagram (which encompasses regularity and \(\pi\)-regularity conditions) are closed under forming semilattices. In fact, we have:

**Proposition 0.1.** Let \(K\) be one of the class of semigroups given in a Table 1 or Table 2. Let \(S\) be a semilattice \(Y\) semigroups \(S_\alpha, \alpha \in Y\). Then \(S\) is a semigroup from class \(K\) if and only if \(S_\alpha\) is in class \(K\), for every \(\alpha \in Y\).

On the other hand, we quote the following known result:

**Theorem 0.2.** A semigroup \(S\) is intra-regular if and only if it is a semilattice of simple semigroups.

So, the classes from the second diagram (which encompasses simplicity and archimedeaness) are not closed under forming semilattices. We want to establish relations between semilattice closures of the classes in the second diagram and the classes in the first diagram.

For undefined notions and notations we refer to [7], [26], [43], [45], or [62].

1. Preliminaries

**Generalized Regular Semigroups**

As a generalization of the concept of an idempotent element, J. von Neumann [1936] introduced the notion of regularity for rings. The class of regular semigroups and its subclasses are the main subject of many books. The authors of some of them are: A. H. Clifford and G. B. Preston [26], M. Petrich [62,63,64], J. M. Howie [45], P. A. Grillet [41], and some others.

An element \(a\) of a semigroup \(S\) is \(\text{regular}\) if \(a \in aSa\). The set of all regular elements of \(S\) is denoted by \(\text{Reg}(S)\) and called \(\text{the regular part of a semigroup}\ \(S\).\) A semigroup \(S\) is \(\text{regular}\) if \(S = \text{Reg}(S)\). Concept of \(\pi\)-regularity, in its various forms appeared first in Ring theory, in paper [53] of McCoy from 1939 (see also [46]). In Semigroup theory, this concept attracted great attention both as a generalization of regularity and a generalization of finiteness and periodicity, and was studied under different names, for example, as: quasi-regularity in the papers of M. Putcha, J. L. Galbiati and M. L. Veronesi; power-regularity initially, and then \(\pi\)-regularity in the papers of S. Bogdanović and S. Milić; eventual regularity in the papers of D. Easdown, R. Edwards and P. Higgins. (See, for example, [4,22,33,34,35,37,43,67].)

An element \(a\) of a semigroup \(S\) is \(\pi\)-\text{regular} if there exists \(n \in \mathbb{N}\) such that \(a^n \in a^nSa^n\), i.e. if some power of \(a\) is in \(\text{Reg}(S)\). A semigroup \(S\) is
\(\pi\)-regular if all of its elements are \(\pi\)-regular. The description of \(\pi\)-regular semigroups given below is from [4].

**Theorem 1.1.** The following conditions on a semigroup \(S\) are equivalent:

(i) \(S\) is \(\pi\)-regular;
(ii) for every \(a \in S\) there exist \(n \in \mathbb{N}\) and \(e \in E(S)\) such that \(R(a^n) = eS\) \(L(a^n) = Se\);
(iii) for every \(a \in S\) there exist \(n \in \mathbb{N}\) such that \(R(a^n)\) \((L(a^n))\) has a left (right) identity.

An element \(a\) of a semigroup \(S\) is intra-regular if \(a \in Sa^2S\). The set of all intra-regular elements of a semigroup \(S\) is denoted by \(\text{Intra}(S)\). A semigroup \(S\) is intra-regular if \(S = \text{Intra}(S)\). Intra-regular semigroups were introduced by R. Croisot, [27]. The result which follows is from that paper.

**Theorem 1.2.** A semigroup \(S\) is intra-regular if and only if it is a semi-lattice of simple semigroups.

Semilattices of simple semigroups are also described in [1, 26, 61, 62]. Chains of simple semigroups were considered in [61] in the following way:

**Theorem 1.3.** A semigroup \(S\) is a chain of simple semigroups if and only if \(a \in SabS\) or \(b \in SabS\), for all \(a, b \in S\).

As a generalization of the previous concept, the notion of an intra-\(\pi\)-regular element was introduced in [67] under the name quasi-intra-regular element. An element \(a\) of a semigroup \(S\) is intra-\(\pi\)-regular if there exists \(n \in \mathbb{N}\) such that \(a^n \in Sa^{2n}S\), i.e. if \(a^n \in \text{Intra}(S)\). A semigroup \(S\) is intra-\(\pi\)-regular if each of its elements is intra-\(\pi\)-regular.

Various concepts of regularity were investigated by R. Croisot in [27], and his study is presented in A. H. Clifford’s and G. B. Preston’s book, [26], as Croisot’s theory. A very important concept in that theory is the notion of left regularity. An element \(a\) of a semigroup \(S\) is left (right) regular if \(a \in Sa^2\) \((a \in a^2S)\). The set of all left (right) regular elements of a semigroup \(S\) is denoted by \(L\text{Reg}(S)\) \((R\text{Reg}(S))\). A semigroup \(S\) is left (right) regular if \(S = L\text{Reg}(S)\) \((S = R\text{Reg}(S))\).

An interesting generalization of the concept of left regularity appears in [2]. So, an element \(a\) of a semigroup \(S\) is left (right) \(\pi\)-regular if there exists \(n \in \mathbb{N}\) such that \(a^n \in Sa^{n+1}\) \((a^n \in a^{n+1}S)\). A semigroup \(S\) is left (right) \(\pi\)-regular if all its elements are left (right) \(\pi\)-regular. The first description of left \(\pi\)-regular semigroups was given in [16].
Theorem 1.4. A semigroup $S$ is left $\pi$-regular if and only if it is intra-$\pi$-regular and $L\text{Reg}(S) = Intra(S)$.

Among various characterizations of left regular semigroups given in [26], there is one about decompositions of these semigroups into a union of left simple semigroups. For decompositions of these semigroups into left simple components in some special cases see [4,61,62,65,66]. Left regular semigroups were described by their left, right and bi-ideals and connections between intersections and products of these ideals in [48,49,50]. As a generalization of completely simple semigroups, we have the following notion: a semigroup $S$ is left (right) completely simple if it is simple and left (right) regular (see [16]). Further, using that notion, left regular semigroups were described as semilattices of left completely simple semigroups. In the theorem which follows, equivalences (i)$\iff$(ii)$\iff$(iii) are from [16], while (iii)$\iff$(iv)$\iff$(v) are from [4] (for some related results one can, also, see [26 - Theorem 6.36]).

Theorem 1.5. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left completely simple;
(ii) $S$ is simple and left $\pi$-regular;
(iii) $S$ is a matrix (right zero band) of left simple semigroups;
(iv) $(\forall a,b \in S) \ a \in S_{ba};$
(v) every principal ideal of $S$ is a left simple subsemigroup of $S$.

We now present the result mentioned above about left regular semigroups (see [16]):

Theorem 1.6. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left regular;
(ii) $S$ is intra-regular and left $\pi$-regular;
(iii) $S$ is a semilattice of left completely simple semigroups;
(iv) $(\forall a,b \in S) \ a \mid b \Rightarrow ab \mid b.$

An element $a$ of a semigroup $S$ is a completely regular or group element if there exists $x \in S$ such that $a = axa$ and $ax = xa$. The set of all completely regular elements of a semigroup $S$ is denoted by $Gr(S)$ and is called the group part of a semigroup $S$. Now we have a well known result:

Lemma 1.7. The following conditions on an element $a$ of a semigroup $S$ are equivalent:
(i) $a \in \text{Gr}(S)$;
(ii) $a \in \text{LReg}(S) \cap \text{RReg}(S)$;
(iii) $a$ is contained in a subgroup of $S$.

A semigroup $S$ is completely regular if $S = \text{Gr}(S)$. Part (iii) from the previous lemma explains why completely regular semigroups are often called unions of groups.

Now we have the following definition: an element $a$ of a semigroup $S$ is completely $\pi$-regular if there exist $x \in S$ and $n \in \mathbb{N}$ such that $a^n = a^n xa^n$ and $a^n x = xa^n$, i.e. if some power of an element $a$ belongs to $\text{Gr}(S)$. A semigroup $S$ is completely $\pi$-regular if each of its elements is completely $\pi$-regular.

The notion of a pseudoinverse of an element, which is essential for completely $\pi$-regular semigroups, is considered by Drazin in [31]. Considerations of completely $\pi$-regular semigroups, i.e. the first investigations of such semigroups, were conducted by W. D. Munn in [60] where they were called pseudo-invertible semigroups. Since that time, with completely $\pi$-regular semigroups as the main object of interest numerous papers have appeared. Various terms have been used for completely $\pi$-regular semigroups in them. Namely, we have: quasi-completely regular semigroups, as well as semigroups in which power of each element lies in a subgroup, in Putsha's papers, [67,69], completely quasi-regular semigroups in the papers of J. L. Galbiati, M. L. Veronesi, [39], groupbound in the papers of B. L. Madison, T. K. Mukherjee and M. K. Sen, [51,52] and P. M. Higgins, [43], quasi periodic at first and then epigroups in the papers of L. N. Shevrin, [75,76], power completely regular in the paper of S. Bogdanović, [4].

**Theorem 1.8.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is completely $\pi$-regular;
(ii) some power of every element of $S$ lies in a subgroup of $S$;
(iii) $(\forall a \in S)(\exists n \in \mathbb{N})$ $a^n \in a^n S a^{n+1}$ ($a^n \in a^{n+1} S a^n$);
(iv) $S$ is $\pi$-regular and left $\pi$-regular;
(v) $S$ is left and right $\pi$-regular;
(vi) every principal bi-ideal of $S$ is $\pi$-regular;
(vii) every left (right) ideal of $S$ is $\pi$-regular.

Idempotent-generating subsemigroup $\langle E(S) \rangle$ of $S$ often belongs to the same class of semigroups as $S$ does.

**Lemma 1.9.** [32] If a semigroup $S$ is (completely) $\pi$-regular then $\langle E(S) \rangle$ is (completely) $\pi$-regular too.
Theorem 1.10. The following conditions on a semigroup S are equivalent:

(i) S completely simple;
(ii) S is simple and completely π-regular;
(iii) S is simple and completely regular;
(iv) S is a matrix of groups;
(v) (∀a, b ∈ S) a ∈ abSa.

The equivalence (i)⇔(ii) from the theorem above is known as Munn’s theorem, (i)⇔(iii) is an immediate consequence of Munn’s theorem, and (i)⇔(iv) is a direct consequence of the definition of Rees matrix semigroup. It is easy to verify that a completely simple semigroup is both left and right completely simple.

A semigroup S isomorphic to the direct product of the group and the left (right) zero band is called left (right) group.

Theorem 1.11. The following conditions on a semigroup S are equivalent:

(i) S is a left group;
(ii) S is a left zero band of groups;
(iii) (∀a, b ∈ S) a ∈ aSb;
(iv) S is regular and E(S) is a left zero band;
(v) S is left simple and contains an idempotent.

Completely regular semigroups or unions of groups were introduced by A. H. Clifford in 1941 (see [25]). The equivalence (i)⇔(ii), from the theorem given below, is one of his results.

Theorem 1.12. The following conditions on a semigroup S are equivalent:

(i) S is completely regular;
(ii) S is a semilattice of completely simple semigroups;
(iii) (∀a ∈ S) a ∈ aSa^2 (a ∈ a^2Sa);
(iv) S is left and right regular;
(v) S is regular and left (right) regular;
(vi) every left ideal of S is a regular semigroup.

Description of chains of completely simple semigroups are from [61].

Theorem 1.13. A semigroup S is a chain of completely simple semigroups if and only if a ∈ abSa or b ∈ baSb, for all a, b ∈ S.

As a consequence of Theorem 1.12., part (i)⇔(ii), we have the following result (see [25]).
Theorem 1.14. A semigroup $S$ is a band if and only if it is a semilattice of matrices (rectangular bands).

ARCHIMEDEAN SEMIGROUPS

Studying the semilattice decompositions of commutative semigroups T. Tamura and N. Kimura in 1954 (see [84]), and, independently of them, G. Thierrin, [87], in the same year, introduced the notion of archimedean semigroups, i.e. semigroups in which for any two elements one of them divides some power of the other.

A semigroup $S$ is archimedean if $a \rightarrow b$, for all $a, b \in S$. A semigroup $S$ is left (right) archimedean if $a \xrightarrow{l} b$ ($a \xrightarrow{r} b$), for all $a, b \in S$. Further, $S$ is t-archimedean if it is both left and right archimedean, i.e. if $a \xrightarrow{t} b$, for all $a, b \in S$. It is clear that an archimedean (left archimedean, right archimedean, t-archimedean) semigroup is a generalization of a simple (left simple, right simple, group) semigroup.

Now, a result which is a consequence of Tamura’s characterization of semilattice-indecomposable semigroups (see [82]) will be given.

Proposition 1.15. An archimedean semigroup is a semilattice-indecomposable.

The previous proposition gives the reason why the study of archimedean semigroups is important. Archimedean semigroups are the most investigated class of semilattice-indecomposable semigroups. The concept of archimedeaness of a semigroup is in close connection with the notion of nil-extension of a semigroup, i.e. with an ideal extension of a semigroup by a nil-semigroup. Namely, the answer of the question: what can we say about archimedean semigroups which belong to the certain classes of semigroups from 1.1. leads to the notion of nil-extensions. The widest such known classes of semigroups are archimedean and intra-$\pi$-regular semigroups, i.e. archimedean semigroups with kernels. In the theorem given below, equivalences (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) are from [67], and (i)$\Leftrightarrow$(ii) is from [7].

Theorem 1.16. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a nil-extension of a simple semigroup;
(ii) $(\forall a, b \in S)(\exists n \in \mathbb{N}) \ a^n \in Sb^{2n}S$;
(iii) $S$ is archimedean and intra-$\pi$-regular;
(iv) $S$ is archimedean and $\text{Intra}(S) \neq \emptyset$.

An archimedean semigroup, in general, need not contain an idempotent.
But, if an archimedean semigroup does contain an idempotent, then, by the previous theorem, it is a nil-extension of a simple semigroup too.

**Theorem 1.17.** A semigroup $S$ is archimedean and contains an idempotent if and only if $S$ is a nil-extension of simple semigroup with an idempotent.

A semigroup $S$ is left (right) completely archimedean if it is archimedean and left (right) $\pi$-regular. This class of semigroups was introduced in [17]. The result which follows is also from that paper.

**Theorem 1.18.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left completely archimedean;
(ii) $S$ is a nil-extension of a left completely simple semigroup;
(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N}) a^n \in Sb^n$.

The class of left archimedean semigroups which are intra-$\pi$-regular coincides with the class of left archimedean semigroups which are left $\pi$-regular (see [7]).

**Theorem 1.19.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a nil-extension of a left simple semigroup;
(ii) $(\forall a, b \in S)(\exists n \in \mathbb{N}) a^n \in Sb^{n+1}$;
(iii) $S$ is left archimedean and left $\pi$-regular.

A semigroup $S$ is completely archimedean if it is archimedean and contains a primitive idempotent. In the theorem given below the equivalence (i)$\iff$(ii) is from [24], the part (ii)$\iff$(iii)$\iff$(iv) is from [22], and (i)$\iff$(v) is from [40].

**Theorem 1.20.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is completely archimedean;
(ii) $S$ is a nil-extension of a completely simple semigroup;
(iii) $S$ is $\pi$-regular and all its idempotents are primitive;
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N}) a^n \in a^nSb^n$ $(a^n \in a^nSb^n)$;
(v) $S$ is archimedean and completely $\pi$-regular.

It is easy to verify that completely archimedean semigroups are both left and right completely archimedean.

What follows now is a description of maximal subgroups of completely archimedean semigroups taken from [22].
Lemma 1.21. Let $S$ be a completely archimedean semigroup. Then, maximal subgroups of $S$ are given by $G_e = cSe$, $e \in E(S)$.

According to Theorem 1.17., an archimedean semigroup with an idempotent need not be completely $\pi$-regular, but for a left archimedean semigroup (see [22]), we have the following:

Lemma 1.22. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left archimedean with an idempotent;
(ii) $S$ is a nil-extension of the left group;
(iii) $S$ is $\pi$-regular and $E(S)$ is a left zero band;
(iv) $(\forall a,b \in S)(\exists n \in N) a^n \in a^nSa^n \Rightarrow (a^n \in ba^nSa^n)$.

A semigroup $S$ is a $\pi$-group if it is $\pi$-regular and contains exactly one idempotent, or equivalently, by T. Tamura’s result, if it is archimedean with exactly one idempotent. The remaining characterizations from the theorem given below are from [10,22].

Theorem 1.23. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a $\pi$-group;
(ii) $S$ is a nil-extension of a group;
(iii) $(\forall a,b \in S)(\exists n \in N) a^n \in b^nSb^n$.

Semilattices of Archimedean Semigroups

Investigations of semigroups which can be decomposed into a semilattice with archimedean components began in [84], where it was proved that any commutative semigroup is a semilattice of archimedean semigroups (commutativity means that components are $t$-archimedean). This result was extended to the class of medial semigroups first in [24], and later to the class of exponential semigroups ([86]). The first complete description of a semigroups which are a semilattice of archimedean semigroups was given by M. S. Putcha ([67]). That is why many authors call such semigroups Putcha’s semigroups. Some other characterizations of these semigroups can be found in [11,21,28,47,62,71,81].

Theorem 1.24. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of archimedean semigroups;
(ii) $S$ is a band of archimedean semigroups;
(iii) $(\forall a,b \in S) a \Rightarrow a^2 \rightarrow b$;
(iv) $(\forall a,b \in S) a^2 \rightarrow ab$;
(v) \((\forall a,b \in S)(\forall n \in \mathbb{N}) a^n \rightarrow ab;\)

(vi) in every homomorphic image of \(S\) containing zero the set of all nilpotent elements is an ideal;

(vii) \((\forall a,b,c \in S) a \rightarrow b \& b \rightarrow c \Rightarrow a \rightarrow c;\)

(viii) \((\forall a,b,c \in S) a \rightarrow c \& b \rightarrow c \Rightarrow ab \rightarrow c;\)

(ix) every bi-ideal of \(S\) is a semilattice of archimedean semigroups;

(x) every one-sided ideal of \(S\) is a semilattice of archimedean semigroups.

In the previous theorem, equivalences (i)\(\Leftrightarrow\)(ii)\(\Leftrightarrow\)(iii) are from [67], (ii)\(\Leftrightarrow\)(iv)\(\Leftrightarrow\)(v) are from [28], (i)\(\Leftrightarrow\)(vi) is from [11], (vii) and (viii) are from [81].

Semilattices of left archimedean and \(t\)-archimedean semigroups have been studied in many papers. In the theorem given below, the equivalence (i)\(\Leftrightarrow\)(ii) is from [72], and (i)\(\Leftrightarrow\)(iii) is from [5] (for the results concerning these semigroups see also [62,67,68]).

**Theorem 1.25.** The following conditions on a semigroup \(S\) are equivalent:

(i) \(S\) is a semilattice of left archimedean semigroups;

(ii) \((\forall a,b \in S) a|b \Rightarrow a \overset{t}{\rightarrow} b;\)

(iii) \((\forall a,b \in S) a \overset{t}{\rightarrow} b.\)

Chains of archimedean semigroups were introduced in [3]. For some results concerning that type of semigroups see also [14].

**Theorem 1.26.** A semigroup \(S\) is a chain of archimedean semigroups if and only if \(ab \rightarrow a\) or \(ab \rightarrow b\), for all \(a, b \in S\).

The rest of this section is devoted to semilattices of archimedean semigroups which are intra-\(\pi\)-regular, left \(\pi\)-regular or completely \(\pi\)-regular. Semilattices of archimedean semigroups which are intra-\(\pi\)-regular were studied for the first time in [67]. In the theorem which follows, the equivalence (i)\(\Leftrightarrow\)(ii) is from that paper, (i)\(\Leftrightarrow\)(iii)\(\Leftrightarrow\)(iv) is from [17], and (i)\(\Leftrightarrow\)(v) is from [18,21].

**Theorem 1.27.** The following conditions on a semigroup \(S\) are equivalent:

(i) \(S\) is a semilattice of nil-extensions of simple semigroups;

(ii) \(S\) is a semilattice of archimedean semigroups and intra-\(\pi\)-regular;

(iii) \((\forall a,b \in S)(\exists n \in \mathbb{N}) (ab)^n \in S(ba)^n(ab)^n S;\)

(iv) \(S\) is an intra-\(\pi\)-regular semigroup and every \(J\)-class of \(S\) which contains an intra-regular element is a subsemigroup;
(v) \( \tau(\mathcal{J}) \) is a semilattice (band) congruence on \( S \).

The semilattices of archimedean semigroups which are left \( \pi \)-regular were introduced and described in [17]. Here is the main result from that paper.

**Theorem 1.28.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is a semilattice of left completely archimedean semigroups;
(ii) \( S \) is a semilattice of nil-extensions of left completely simple semigroups;
(iii) \( S \) is a semilattice of archimedean semigroups and left \( \pi \)-regular;
(iv) \( (\forall a, b \in S)(\exists n \in \mathbb{N}) \ (ab)^n \in S a (ab)^n \);
(v) \( S \) is a left \( \pi \)-regular semigroup and every \( L(J) \)-class of \( S \) which contains a left regular element is a subsemigroup.

The semigroups described in the previous theorem can be treated as a generalization of the semilattices of nil-extensions of left simple semigroups (introduced in [67], and also characterized in [7,13]).

**Theorem 1.29.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is a semilattice of nil-extensions of left simple semigroups;
(ii) \( S \) is a semilattice of left archimedean semigroups and left \( \pi \)-regular;
(iii) \( (\forall a, b \in S)(\exists n \in \mathbb{N}) \ (ab)^n \in S a^2 (ab)^n + 1 \).

A semigroup \( S \) is uniformly \( \pi \)-regular if it is a semilattice of completely archimedean semigroups. Studies of decompositions of semigroups into semilattices of completely archimedean semigroups began in L. N. Shevrin’s papers. The final results of his several year long investigations are given in [75]. Similar results concerning decompositions of completely \( \pi \)-regular semigroups into semilattices of archimedean semigroups were obtained by J. L. Galbiati and M. L. Veronesi in [38], where they started such investigations, and by M. L. Veronesi in paper [88], where it ended. The main result is

**Theorem 1.30.** Shevrin-Veronesi. A semigroup \( S \) is uniformly \( \pi \)-regular (a semilattice of completely archimedean semigroups) if and only if \( S \) is \( \pi \)-regular and \( Reg(S) = Gr(S) \).

Some other characterizations of semilattices of completely archimedean semigroups, in general and some special cases, can be found in papers of S. Bogdanović, and S. Bogdanović and M. Ćirić (see, for example, [7,13,17]).
Theorem 1.31. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a uniformly $\pi$-regular semigroup;
(ii) $S$ is $\pi$-regular and $\text{Reg}(S) \subseteq L\text{Reg}(S)$;
(iii) $S$ is a semilattice of archimedean semigroups and completely $\pi$-regular;
(iv) $(\forall a,b \in S)(\exists n \in \mathbb{N}) (ab)^n \in (ab)^n b S(ab)^n$;
(v) $S$ is completely $\pi$-regular and $(\forall a \in S)(\forall e \in E(S)) a|e \Rightarrow a^2|e$;
(vi) $S$ is completely $\pi$-regular and $(\forall a, b \in S)(\forall e \in E(S)) a|e$ and $b|e \Rightarrow ab|e$;
(vii) $S$ is completely $\pi$-regular and $(\forall e, f, g \in E(S)) e|g$ and $f|g \Rightarrow ef|g$;
(viii) $S$ is a $\pi$-regular semigroup and every $\mathcal{L}(J)$ of $S$ which contains an idempotent is a subsemigroup of $S$;
(ix) $S$ is a semilattice of left completely archimedean semigroups and $\pi$-regular.

In the previous theorem, equivalences (i)$\iff$(iii)$\iff$(v) are from [67], (i)$\iff$(iv) is from [8], (i)$\iff$(vi) are from [75], and (i)$\iff$(ii)$\iff$(ix) are from [17].

Chains of completely archimedean semigroups were introduced and described in [5]. In the theorem given below, the equivalence (i)$\iff$(ii) is from [5], and (i)$\iff$(iii) is from [15].

Theorem 1.32. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a chain of completely archimedean semigroups;
(ii) $S$ is completely $\pi$-regular and $e \in efS$ or $f \in feS$, for all $e, f \in E(S)$;
(iii) $S$ is $\pi$-regular and $\text{Reg}(S)$ is a chain of completely simple semigroups.

The rest of this section is devoted to two very important subclasses of the class of semigroups considered in Theorem 1.31. Semilattices of nil-extensions of left groups were introduced in [67]. Descriptions of these semigroups given here are from [5,6,8,76].

Theorem 1.33. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of nil-extensions of left groups;
(ii) $S$ is a semilattice of left archimedean semigroups and $\pi$-regular;
(iii) $(\forall a,b \in S)(\exists n \in \mathbb{N}) (ab)^n \in (ab)^n S(ba)^n$;
(iv) $(\forall a,b \in S)(\exists n \in \mathbb{N}) (ab)^n S a^{2n}$. 
(v) $S$ is uniformly $\pi$-regular and left $\pi$-inverse;
(vi) $S$ is uniformly $\pi$-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (efe)^n$;
(vii) $S$ is $\pi$-regular and $a = axa$ implies $ax = xa^2x$;
(viii) $S$ is a completely $\pi$-regular semigroup and every $J$-class which contains an idempotent is a left group.

Some other characterizations of the semigroups from Theorem 1.33. can be found in [9,15].

The semilattices of $\pi$-groups, i.e. the semilattices of nil-extensions of groups were introduced in [67]. The results concerning these semigroups can also be found in [5,8,36,39,73,76,88].

**Theorem 1.34.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of nil-extensions of groups;
(ii) $S$ is uniformly $\pi$-regular and $\pi$-inverse;
(iii) $S$ is uniformly $\pi$-regular and for all $e, f \in E(S)$ there exists $n \in \mathbb{N}$ such that $(ef)^n = (fe)^n$;
(iv) $S$ is $\pi$-regular and $a = axa$ implies $ax = xa$;
(v) $S$ is a semilattice of $t$-archimedean semigroups and $\pi$-regular;
(vi) $(\forall a, b \in S)(\exists n \in \mathbb{N})(ab)^n \in b^{2^n}Sa^{2^n}$;
(vii) $S$ is a $\pi$-regular semigroup and every $J$-class which contains an idempotent is a group.

In the previous theorem, the equivalences $(i) \iff (ii) \iff (iii)$ are from [88], $(i) \iff (iv)$ is from [5], $(i) \iff (vi)$ is from [8], and $(i) \iff (vii)$ is from [76].

2. Semilattices of Nil-extensions of Simple Semigroups

Studying minimal conditions on semigroups, E. Hotzel, [44], introduced weakly periodic, or as we call them, $\pi$-semisimple semigroups. It is easy to see that $\pi$-semisimple semigroups can be treated as a generalization of semisimple ones. According to the decomposability of $\pi$-semisimple semigroups into semilattices of archimedean semigroups, it will be shown that such semigroups coincide with intra-$\pi$-regular semigroups which are semilattices of archimedean semigroups. Apart from that result we will give some other criteria in the term of equality of some subsets of such semigroups.

An element $a$ of a semigroup $S$ is semisimple if $a \in SaSaS$. A set of all semisimple elements of $S$ will be denoted by $\text{Semis}(S)$. A semigroup $S$
is semisimple if $S = \text{Semis}(S)$. According to the results from [58,74,77], a semigroup $S$ is semisimple if none of its principal factors are null, or, equivalently, if $I = I^2$, for every ideal $I$ of $S$. The importance of the previous concept can also be seen from [59]. A semigroup $S$ is $\pi$-semisimple, if every $a \in S$ has a power $a^n$, $n \in \mathbb{N}$, satisfying $a^n \in \text{Semis}(S)$. The next result is obvious:

**Lemma 2.1.** A semigroup $S$ is $\pi$-semisimple if and only if for every $a \in S$ there exists $n \in \mathbb{N}$ such that $J^2(a^n) = J(a^n)$.

Now, we will give connections between the existence of semisimple and intra-regular elements.

**Lemma 2.2.** For a semigroup $S$ we have $\text{Semis}(S) \neq \emptyset$ if and only if $\text{Intra}(S) \neq \emptyset$.

A characterization of semisimple semigroups which follows is in a great deal a consequence of the previous lemma.

**Lemma 2.3.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is semisimple;
(ii) $(\forall a \in S) ~ J^2(a) = J(a)$;
(iii) $J(a)$ is generated by an intra-regular element for every $a \in S$.

Now, we will consider archimedean semigroups which are $\pi$-semisimple. In fact, we will show that these semigroups coincide with already known nil-extensions of simple semigroups.

**Lemma 2.4.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a nil-extension of a simple semigroup;
(ii) $S$ is archimedean and is $\pi$-semisimple;
(iii) $S$ is archimedean and there exists $b \in S$ such that $b \in Sb^kS$, for every $k \in \mathbb{N}$;
(iv) $\forall a, b \in S)(\exists n \in \mathbb{N}) ~ a^n \in Sb^nS$.

Let us introduce Green’s subsets of a semigroup $S$. Let $T$ be one of well-known Green’s relations on a semigroup $S$. Then by $U_T(S)$ we will denote the union of all $T$-classes of $S$ which are subsemigroups of $S$, by $L^2_T(S)$ the union of all $T$-classes of $S$ which are left simple subsemigroups of $S$, by $L_T(S)$ the union of all $T$-classes of $S$ which are left groups and by $H_T(S)$ the union of all $T$-classes of $S$ which are groups.

Semigroups which are semilattices of nil-extensions of simple semigroups, or, intra-$\pi$-regular semigroups which are decomposed into
archimedean components, admit a great number of different characterizations (see, Theorem 1.27.). It is an easy-to-prove consequence of Lemma 2.4. and Proposition 0.1. that these semigroups coincide with semilattices of archimedean semigroups which are \(\pi\)-semisimple. The main result of this section gives us a connections between decomposability of \(\pi\)-semisimple (intra-\(\pi\)-regular) semigroups into archimedean components and certain equalities of (generalized) regular and Green’s subset of such a semigroup.

**Theorem 2.5.** The following conditions on a semigroup \(S\) are equivalent:

1. \(S\) is a semilattice of nil-extensions of simple semigroups;
2. \(S\) is intra \(\pi\)-regular and \(\text{Intra}(S) = \mathcal{U}_\mathcal{J}(S)\);
3. \(S\) is \(\pi\)-semisimple and \(\text{Semis}(S) = \mathcal{U}_\mathcal{J}(S)\).

Semigroups with a non-empty set of idempotents which are semilattices of semigroups with kernels were studied in [67]. Here, the description of these semigroups is given with the help of Tamura’s \(cm\)-property (see [81]) applied to idempotents.

**Theorem 2.6.** Let \(E(S) \neq \emptyset\). Then the following conditions on a semigroup \(S\) are equivalent:

1. \((\forall a \in S)(\forall e \in E(S)) a|e \Rightarrow a^2|e;\)
2. \((\forall a, b \in S)(\forall e \in S) a|e \& b|e \Rightarrow ab|e;\)
3. \((\forall e, f, g \in E(S)) e|g \& f|g \Rightarrow ef|g;\)
4. \(S\) is a semilattice \(Y\) of semigroups \(S_\alpha, \alpha \in Y\), where \(S_\alpha\) has a kernel \(K_\alpha\) such that \(E(S_\alpha) \subseteq K_\alpha\) or \(E(S_\alpha) = \emptyset\), for an arbitrary \(\alpha \in Y\).

### 3. Semilattices of Left Strongly Archimedean Semigroups

Within the class of \(\pi\)-semisimple semigroups, mentioned in the previous section, two interesting subclasses will be taken into consideration. First, we have left quasi-regular semigroups introduced by J. Calais, [23]. Second, as a generalization of these semigroups, we introduce and describe left quasi-\(\pi\)-regular semigroups. In order to describe left quasi-regular and intra-regular semigroups we will introduce the notion of left strongly simple semigroups, we will give various characterizations of these semigroups, and then describe intra-regular and left quasi-regular semigroups as semilattices of left strongly simple semigroups. Analogously, we will introduce the notion of left strongly archimedean semigroup, and finally, we will prove
that left strongly archimedean semigroups are precisely archimedean components in semilattice decompositions of left quasi-$\pi$-regular semigroups.

An element $a$ of a semigroup $S$ is a left (right, completely) quasi-
regular if $a \in SaS$ ($a \in aSa$, $a \in aSa \cap SaS$), for all $a \in S$. A set of all left (right, completely) quasi-regular elements of $S$ will be denoted by $LQReg(S)$ ($RQReg(S)$, $QGr(S)$). A semigroup $S$ is left (right, completely) quasi-
regular if $LQReg(S) = S$ ($RQReg(S) = S$, $QGr(S) = S$). Here, as the generalization of the previous concept we have the following notion: a semigroup $S$ is left (right) quasi-$\pi$-regular if for every $a \in S$ some power of $a$ belongs to $LQReg(S)$ ($RQReg(S)$). A semigroup $S$ is completely quasi-
$\pi$-regular if it is both left and right quasi-$\pi$-regular. The structure of left quasi-$\pi$-regular semigroups is given in the next theorem.

**Theorem 3.1.** The following conditions on a semigroup $S$ are equivalent:

- $S$ is left quasi-$\pi$-regular;
- $(\forall a \in S)(\exists n \in \mathbb{N}) L^2(a^n) = L(a^n);$
- $S$ is $\pi$-semisimple and $Semis(S) = LQReg(S)$.

Now, connections between the existance of left quasi-regular and left
regular elements will be given.

**Lemma 3.2.** For a semigroup $S$ we have that $LReg(S) \neq \emptyset$ if and only if $LQReg(S) \neq \emptyset$.

Using the previous lemma and Theorem 3.1., the descriptions of left
regular semigroups can be obtained.

**Lemma 3.3.** The following conditions on a semigroup $S$ are equivalent:

- (i) $S$ is a left quasi-regular semigroups;
- (ii) $(\forall a \in S) L^2(a) = L(a);$
- (iii) $L(a)$ is generated by left regular element for every $a \in S$.

As the consequence of the previous lemma and its dual we can describe
completely quasi-regular semigroups as the ones with idempotent (principal) one-sided ideals. As a consequence of Theorem 3.1. and its dual we will give the result concerned with semigroups which are the generalization of completely quasi-regular ones.

**Corollary 3.4.** The following conditions on a semigroup $S$ are equivalent:

- (i) $S$ is completely quasi-$\pi$-regular;
- (ii) $S$ is left quasi-$\pi$-regular and $Semis(S) = RQReg(S);$
(iii) $S$ is $\pi$-semisimple and $\text{Semis}(S) = LQReg(S) = RQReg(S)$;
(iv) $(\forall a \in S)(\exists n \in \mathbb{N}) a^n \in Sa^nSa^n \cap a^nSa^nS$.

Next, let us turn our attention to the simple semigroups and their semilattices, i.e., according to Theorem 1.2., intra-regular semigroups which belong to the classes of semigroups which have just been described. First, it is convenient to introduce the following terminology: a semigroup $S$ is a left (right) strongly simple semigroup if $S$ is simple and left (right) quasi-regular. A semigroup $S$ is a strongly simple semigroup if $S$ is both left and right strongly simple.

Lemma 3.5. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left strongly simple;
(ii) $(\forall a, b \in S) a \in SbSa$;
(iii) every left ideal of $S$ is a simple semigroup;
(iv) $S$ is simple and every left ideal of $S$ is idempotent;
(v) $S$ is simple and every left ideal of $S$ is an intra-regular semigroup;
(vi) $S$ is simple and left quasi-$\pi$-regular semigroup.

The equivalence (i)$\iff$(vi) from Lemma 3.5. can be viewed as generalization of the well known Munn’s theorem (see [26, Theorem 2.55]).

From the previous lemma and its dual we can easily deduce that strongly simple semigroups coincide with simple and completely quasi-(\pi-)regular semigroups. The next characterizations of strongly simple semigroups, given in terms of their ideals and elements, are consequences of the previous lemma, too.

Corollary 3.6. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is strongly simple;
(ii) $(\forall a, b \in S) a \in SbSa \cap aSbS$;
(iii) every one-sided ideal of $S$ is a simple semigroup;
(iv) $S$ is simple and every one-sided ideal of $S$ is idempotent;
(v) $S$ is simple and every one-sided ideal of $S$ is intra-regular.

Now, we will describe left quasi-regular and intra-regular semigroups from the perspective of the results given in Lemma 3.5. and Proposition 0.1.

Theorem 3.7. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of left strongly simple semigroups;
(ii) $(\forall a \in S) a \in Sa^2Sa$;
(iii) every left ideal of $S$ is an intra-regular subsemigroup of $S$;
(iv) every left ideal of $S$ is a semisimple subsemigroup of $S$.

We can give a description of a semilattice of strongly simple semigroups, i.e. intra-regular and completely quasi-(π-)regular semigroups, as an easy consequence of Theorem 3.7. and its dual.

**Corollary 3.8.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of strongly simple semigroups;
(ii) $(\forall a \in S)\ a \in S a^2 S a \cap a S a^3 S$;
(iii) every one-sided ideal of $S$ is an intra-regular subsemigroup of $S$;
(iv) every one sided ideal of $S$ is a semisimple subsemigroup of $S$.

Within the class of semigroups considered in Theorem 3.7. there is a subclass of semigroups which is a generalization of those from [61].

**Theorem 3.9.** A semigroup $S$ is a chain of left strongly simple semigroups if and only if $a \in S a b S a$ or $b \in S a b S b$, for all $a, b \in S$.

At the end of this section we will give results concerning archimedean semigroups which are left quasi-π-regular and their semilattices. For that purpose we introduce the following notion: a semigroup $S$ is left (right) strongly archimedean if $S$ is archimedean and left (right) quasi-π-regular.

A semigroup $S$ is strongly archimedean if $S$ is both left and right strongly archimedean.

**Theorem 3.10.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is left strongly archimedean;
(ii) $S$ is a nil-extension of a left strongly simple semigroup;
(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in S b S a^n$;
(iv) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in S b^n S a^n$.

Using Theorem 3.10. we can describe left quasi-π-regular semigroups which are decomposable into archimedean components.

**Theorem 3.11.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of left strongly archimedean semigroups;
(ii) $S$ is a semilattice of archimedean semigroups and left quasi-π-regular;
(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in S a^2 S (ab)^n$;
(iv) $S$ is left quasi-π-regular and \( \text{Semis}(S) = U_\mathcal{J}(S) \);
(v) $S$ is intra $\pi$-regular and \( \text{LQReg}(S) = \text{Intra}(S) = U_\mathcal{J}(S) \);
(vi) $S$ is intra-$\pi$-regular and every $J$-class of $S$ containing an intra-regular element is a left strongly simple semigroup.

Because of Theorem 3.10. and its dual, strongly archimedean semigroups are nil-extensions of strongly simple semigroups. That fact as well as the previous theorem and its dual are used for characterizing semilattices of strongly archimedean semigroups.

**Corollary 3.12.** The following conditions on a semigroup $S$ are equivalent:

1. $S$ is a semilattice of strongly archimedean semigroups;
2. For all $a, b \in S$, there exists an $n \in \mathbb{N}$ such that $(ab)^n \in S\langle a \rangle^2S \cap (ab)^n S\langle a \rangle^2S$;
3. $S$ is completely quasi-$\pi$-regular and $\text{Semis}(S) = \bigcup J(S)$;
4. $S$ is intra-$\pi$-regular and $\text{QGr}(S) = \text{Intra}(S) = \bigcup J(S)$;
5. $S$ is intra-$\pi$-regular and every $J$-class of $S$ containing an intra-regular element is a strongly simple semigroup.

4. **Semilattices of Left Completely Archimedean Semigroups**

Left $\pi$-regular, left completely simple, left regular, left completely archimedean semigroups and their semilattices are already known types of semigroups (see Section 1). Here, in this section, we considered them from the perspective of the results from Sections 2 and 3. First, some connections of left $\pi$-regular semigroups with left quasi-$\pi$-regular ones will be given.

**Theorem 4.1.** The following conditions on a semigroup $S$ are equivalent:

1. Every bi-ideal of $S$ is left quasi-$\pi$-regular;
2. Every left ideal of $S$ is left quasi-$\pi$-regular;
3. $S$ is left $\pi$-regular.

Concerning connections of left completely simple semigroups with the left strongly simple ones from the previous section we have the next:

**Lemma 4.2.** A semigroup $S$ is left completely simple if and only if every left ideal of $S$ is a left strongly simple subsemigroup of $S$.

Left regular semigroups are described by their left ideals through the membership in certain classes of semigroups, i.e. by their connection with left quasi-regular semigroups.

**Lemma 4.3.** A semigroup $S$ is left regular if and only if every left ideal of $S$ is left quasi-regular.
It is known that left regular semigroups coincide with semilattices of left completely simple semigroups (see Theorem 1.6.). Here, we distinguish chains of left completely simple semigroups.

**Theorem 4.4.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a chain of left completely simple semigroups;
(ii) $(\forall a, b \in S) \ a \in Sba \ or \ b \in Sb^2$;
(iii) $(\forall a, b \in S) \ a \in Sab^2 \ or \ b \in Sab^2$.

Archimedean semigroups which are left $\pi$-regular, i.e. left completely archimedean semigroups and their connection with left strongly archimedean semigroups are given in the following lemma.

**Lemma 4.5.** A semigroup $S$ is left completely archimedean if and only if every left ideal of $S$ is a left strongly archimedean semigroup.

Now, at the end of this section, we turn our attention to the left $\pi$-regular semigroups decomposable into archimedean components (see Theorem 1.28.). Namely, we have left $\pi$-regular semigroups with certain equalities between their (generalized) regular, on the one hand, and Green’s subsets on the other.

**Theorem 4.6.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of left completely archimedean semigroups;
(ii) $S$ is left $\pi$-regular and $LReg(S) = U_C(S) = U_J(S)$;
(iii) $S$ is left $\pi$-regular and $Semis(S) = U_J(S)$;
(iv) every left ideal of $S$ is a semilattice of left strongly archimedean semigroups.

According to Theorem 1.19. we have that the class of left archimedean semigroups which are intra-$\pi$-regular coincides with the class of left archimedean semigroups which are left $\pi$-regular, i.e. we have the class of nil-extensions of left simple semigroups. Taking into account the results from Section 2 and Section 3 we end this section with some other characteristics of nil-extensions of left simple semigroups and their semilattices.

**Lemma 4.7.** The following conditions on a semigroup $S$ are equivalent

(i) $S$ is a nil-extension of a left simple semigroup;
(ii) $(\forall a, b \in S)(\exists n \in N) \ a^n \in Sb^n$;
(iii) $S$ is left archimedean and left quasi $\pi$-regular.

**Theorem 4.8.** The following conditions on a semigroup $S$ are equivalent:
5. Semilattices of Nil-extensions of Simple and Regular Semigroups

The concept of regularity as well as its generalization, concept of \( \pi \)-regularity, in their various forms appeared first in the Ring theory. These concepts have awaken enormous attention among the specialists in semigroup theory, as being evidence by a number of monographies and papers (see, for example [25,26,41,42,45,49,62,63] for regular, and [4,7,13,38] for \( \pi \)-regular semigroups). Using results from the previous sections we will give some other results concerning \( \pi \)-regular, regular, regular and simple, regular and intra-regular, semilattices of nil-extensions of simple and regular semigroups.

**Lemma 5.1.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is \( \pi \)-regular;
(ii) \( \forall a, b \in S) (\exists n \in \mathbb{N}) (ab)^n \in Sa^{2n}; \)
(iii) \( S \) is \( \pi \)-semisimple and \( \text{Semis}(S) = L_\pi^S(S) \);
(iv) \( S \) is a left quasi-\( \pi \)-regular semigroup and \( LQ\text{Reg}(S) = L_\pi^S(S). \)

**Lemma 5.2.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is left (right) quasi-regular and \( \pi \)-regular;
(ii) \( S \) is semisimple and \( \pi \)-regular;
(iii) \( S \) is regular.

Talking about regular and simple semigroups, we will see that \( \pi \)-regularity on simple semigroups goes down to the "ordinary" regularity.

**Lemma 5.3.** The following conditions on a semigroup \( S \) are equivalent:

(i) \( S \) is regular and simple;
(ii) \( S \) is \( \pi \)-regular and simple;
(iii) \( (\forall a, b \in S) a \in aSbSa; \)
(iv) \( S \) is regular and left (right) strongly simple;
(v) every bi-ideal of \( S \) is a simple semigroup.

Regular and intra-regular semigroups were studied in many papers (for example, see [49,50,67]). Characterizations of these semigroups given here are consequences of the previous results.
Theorem 5.4. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of simple and regular semigroups;
(ii) $(\forall a \in S) \ a \in aSa^2Sa$;
(iii) every bi-ideal of $S$ is an intra-regular subsemigroup of $S$;
(iv) every bi-ideal of $S$ is a semisimple subsemigroup of $S$;
(v) every left ideal of $S$ is right quasi-regular;
(vi) $S$ is $\pi$-regular and intra-regular.

Theorem 5.5. A semigroup $S$ is a chain of simple and regular semigroups if and only if $a \in aSabSa$ or $b \in bSabSb$, for every $a, b \in S$.

Archimedean and $\pi$-regular semigroups, as it is shown below, are in close connection with the structure of simple and regular semigroups.

Theorem 5.6. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is $\pi$-regular and archimedean;
(ii) $S$ is a nil-extension of a simple and regular semigroup;
(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N}) \ a^n \in a^nSbSa^n$.

Now, we can give the main result of this section, i.e. we can describe $\pi$-regular semigroups which are decomposable into semilattices of archimedean semigroups.

Theorem 5.7. The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of nil-extensions of simple and regular semigroups;
(ii) $S$ is $\pi$-regular and a semilattice of archimedean semigroups;
(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N}) \ (ab)^n \in (ab)^nSa^2S(ab)^n$;
(iv) $S$ is $\pi$-regular and $(\forall a \in S)(\forall e \in E(S)) \ a|e \Rightarrow a^2|e$;
(v) $S$ is $\pi$-regular and $(\forall a, b \in S)(\forall e \in E(S)) \ a|e$ and $b|e \Rightarrow ab|e$;
(vi) $S$ is $\pi$-regular and $(\forall e, f, g \in E(S)) \ e|g$ and $f|g \Rightarrow ef|g$;
(vii) $S$ is intra-$\pi$-regular and each $J$-class of $S$ containing an intra-regular element is a regular subsemigroup of $S$;
(viii) $S$ is $\pi$-regular and each $J$-class of $S$ containing an idempotent is a subsemigroup of $S$;
(ix) $S$ is $\pi$-regular and $\tau(J)$ is a semilattice (or a band) congruence on $S$;
(x) $S$ is a semilattice of nil-extensions of simple semigroups and $\text{Intra}(S) = \text{Reg}(S)$;
(xi) $S$ is $\pi$-regular and $\text{Reg}(S) = \text{Intra}(S) = \cup J(S)$. 
Within the class of semigroups mentioned in the previous theorem, chains of archimedean and π-regular semigroups will be considered.

**Theorem 5.8.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a chain of nil-extensions of simple and regular semigroups;
(ii) $(\forall a,b \in S) (\exists n \in \mathbb{N}) a^n \in a^nSabSa^n$ or $b^n \in b^nSabSb^n$;
(iii) $S$ is π-regular and $(\forall e,f \in E(S)) ef|e$ or $ef|f$;
(iv) $S$ is π-regular and $\text{Reg}(S)$ is a chain of simple and regular semigroups.

6. Semilattices of Completely Archimedean Semigroups

The class of completely π-regular semigroups and its subclasses consist of: completely simple, completely regular, completely archimedean semigroups or semilattices of completely archimedean semigroups are the main subject of this section. Of course, they are described from the perspective of the results from Sections 2-5.

**Theorem 6.1.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is completely π-regular;
(ii) every bi-ideal of a semigroup $S$ is completely quasi-π-regular;
(iii) $S$ is left π-regular and right quasi-π-regular.

**Lemma 6.2.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is completely simple;
(ii) $S$ is left (right) completely simple and has an idempotent;
(iii) $S$ is simple, left regular and right quasi-regular;
(iv) $S$ is simple, left regular and right quasi-π-regular.

Completely regular smigroups (or union of groups) are considered by using results from Theorem 6.1. and Lemma 6.2.

**Theorem 6.3.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a completely regular semigroup;
(ii) every left ideal of $S$ is right regular;
(iii) every left ideal of $S$ is completely quasi-regular;
(iv) every bi-ideal of $S$ is left quasi-regular;
(v) $S$ is left regular and right quasi-regular;
(vi) $S$ is left regular and right quasi-π-regular.
Archimedean semigroups which are completely $\pi$-regular, i.e. completely archimedean semigroups are the most described subclass of the class of archimedean semigroups (see, for example, Theorem 1.20.). Here we have the following result.

**Theorem 6.4.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is completely archimedean;
(ii) $S$ is $\pi$-regular and $eSe$ is a unipotent monoid for every $e \in E(S)$;
(iii) $S$ is $\pi$-regular and $E(eS)$ is a semigroup of right zeros for every $e \in E(S)$;
(iv) $S$ is completely $\pi$-regular and $(E(S))$ is a (completely) simple semigroup;
(v) $S$ is left completely archimedean and contains an idempotent.

There are many results concerning completely $\pi$-regular semigroups which are decomposable into semilattices of archimedean components (see Theorem 1.31). We will describe semilattices of completely archimedean semigroups by certain equalities between the group part and Green's sub-
sets of completely $\pi$-regular semigroups.

**Theorem 6.5.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a semilattice of completely archimedean semigroups (uniformly $\pi$-regular);
(ii) $S$ is completely $\pi$-regular, $\text{Reg}((E(S)) = \text{Gr}((E(S))$ and for all $e, f \in E(S)$, $f|e$ in $S$ implies $f|e$ in $(E(S))$;
(iii) $S$ is completely $\pi$-regular and $\text{Gr}(S) = \mathcal{U}_J(S) = \mathcal{U}_L(S)$;
(iv) every left ideal of $S$ is a semilattice of nil-extensions of simple and regular semigroups;
(v) every bi-ideal of $S$ is a semilattice of strongly archimedean semigroups.

Chains of completely archimedean semigroups were studied in [14,15]. Here characterizations of these semigroups are given in terms of their idempotents.

**Theorem 6.6.** The following conditions on a semigroup $S$ are equivalent:

(i) $S$ is a chain of completely archimedean semigroups;
(ii) $S$ is completely $\pi$-regular and $e \in e\langle E(S)\rangle fe$ or $f \in f\langle E(S)\rangle ef$, for all $e, f \in E(S)$;
(iii) $S$ is completely $\pi$-regular and $e \in efE(S) \text{ or } f \in (E(S))ef$, for all $e, f \in E(S)$;
(iv) $S$ is a completely $\pi$-regular semigroup and $\langle E(S) \rangle$ is a chain of completely simple semigroups.

**Corollary 6.7.** A semigroup $S$ is a semilattice of nil-extensions of left groups if and only if $S$ is completely $\pi$-regular and $\overline{\text{Gr}}(S) = L_{\mathcal{F}}(S)$.

**Corollary 6.8.** A semigroup $S$ is a semilattice of nil-extensions of groups if and only if $S$ is completely $\pi$-regular and $\overline{\text{Gr}}(S) = H_{\mathcal{F}}(S)$.

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